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Homomorphisms of convolution algebras

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Abstract

We establish an explicit, algebraic, one-to-one correspondence between the $*$-homomorphisms, $\varphi : L^1(F) \to M(G)$, of group and measure algebras over locally compact groups F and G, and group homomorphisms, $\phi : F \to \mathbb{M}_\phi$, where $\mathbb{M}_\phi$ is a semi-topological subgroup of $(M(G), w^*)$. We show how to extend any such $*$-homomorphism to a larger convolution algebra to obtain nicer continuity properties. We augment Greenleaf's characterization of the contractive subgroups of $M(G)$ (Greenleaf, 1965 [17]) by completing the description of their topological structures. We show that not every contractive homomorphism has the dual form of Cohen's factorization in the abelian case, thus answering a question posed by Kerlin and Pepe (1975) in [27]. We obtain an alternative factorization of any contractive homomorphism $\varphi : L^1(F) \to M(G)$ into four homomorphisms, where each of the four factors is one of the natural types appearing in the Cohen factorization.

Keywords: Locally compact group; Group algebra; Measure algebra; Homomorphism; Introverted subspace

Let F and G be locally compact groups. The convolution "homomorphism problem" in abstract harmonic analysis asks for a description of all bounded homomorphisms between group and measure algebras, $L^1(F)$ and $M(G)$ (and related convolution algebras); the dual version of the homomorphism problem asks for a description of all homomorphisms between Fourier and Fourier–Stieltjes algebras, $A(F)$ and $B(G)$. Wendel's theorem, which describes the isometric isomorphisms between group algebras $L^1(F)$ and $L^1(G)$ [40], is among the first significant contributions towards a solution to this old problem. When F and G are

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abelian, Paul Cohen solved (both versions of) the homomorphism problem in 1960 [2,34]. Since the publication of [2], many mathematicians have worked towards extending Cohen's beautiful theorem to the nonabelian setting and in 1965, Greenleaf [17] successfully provided a characterization all contractive homomorphisms $\varphi : L^1(F) \to M(G)$ for arbitrary groups. As noted by Kerlin and Pepe [27], this characterization is less tractable than the one obtained by Cohen in the abelian case. Thus, beyond the general convolution homomorphism problem, an open problem has asked for a factorization of contractive homomorphisms that more closely shares the spirit of Cohen's theorem. Over the years this last question has been answered when either F or G is assumed to be abelian, some simplifications to Greenleaf's original arguments have been obtained, and (isometric) isomorphisms of convolution algebras have been intensely investigated – for example, see [25,38,27,26,31,13,14, 41,15,16,11,5]. However, since the publication of [17], very little progress has been made with the general version of the convolution homomorphism problem.

In the dual situation, isometric isomorphisms of Fourier and (reduced) Fourier–Stieltjes algebras are characterized in [39] and [29]. (The isometric isomorphism theorems of Wendel, Johnson, Strichartz, and Walter are extended to the situation of Kac algebras in [6]; also see [7].) M. Ilie and N. Spronk employed the operator space structures of $A(F)$ and $B(G)$ to extend Cohen's theorem, and an intermediate result due to Host [20], to the dual setting when F is amenable [21] and [22]. The dual version of the homomorphism problem – which in general also remains open – has since enjoyed a revived period of investigation; for example, see [23,24,33]. In particular, H.L. Pham recently showed that every contractive homomorphism $\varphi : A(F) \to B(G)$ factors as $\varphi = l_{r_0} \circ s \circ j_\theta \circ l_{u_0}$ where $l_r u(s) = u(rs)$; $\theta : G_0 \to F$ is a continuous homomorphism, or anti-homomorphism, defined on an open subgroup G_0 of G and $j_\theta u = u \circ \theta$; and $s : B(G_0) \to B(G)$ is the expansion homomorphism given by $s(u)(h) = u(h)$ if $h \in G_0$, $s(u)(h) = 0$ otherwise [33, Theorem 5.1]. This extends Cohen's characterization of all homomorphisms $\varphi : A(F) \to B(G)$ in the abelian setting and is also consistent with Ilie's and Spronk's work.

$$
\begin{tikzcd}
A(F) \arrow[rr, "\varphi"] \arrow[d, "l_{u_0}"'] & & B(G) \\
A(F) \arrow[r, "j_\theta"] & B(G_0) \arrow[r, "s"] & B(G) \arrow[u, "l_{r_0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L^1(F) \arrow[rr, "\varphi"] \arrow[d, "A_\alpha"'] & & M(G) \\
L^1(F) \arrow[r, "j^*_{\theta_K}"] & M(G/K) \arrow[r, "S^*_K"] & M(G) \arrow[u, "A_{\rho_0}"']
\end{tikzcd}
\qquad (0.1)
$$

Assuming that F and G are abelian, one can check that the precise dual form of the homomorphisms l_r, j_θ and s are the maps $A_\alpha = M^*_\alpha$, $j^*_{\theta_K}$ and S^*_K – defined precisely in Section 5 – where α is a continuous character, $\theta_K : F \to G/K$ is a continuous homomorphism and K is a compact subgroup of G. An open problem, posed explicitly in [27], thus asks whether every contractive homomorphism $\varphi : L^1(F) \to M(G)$ has the factorization illustrated in the second of the above diagrams when F and G are not assumed to be abelian. Kerlin and Pepe answered this question affirmatively in the case when G is abelian (and a different proof of this fact is given in Section 5). With Example 5.4 we will give a negative answer to this question, in general. Our main result of Section 5, Theorem 5.11, provides an alternative factorization of any contractive homomorphism φ into four, canonically defined, homomorphisms, $\varphi = j^*_{\theta_H} \circ A_{\alpha_{\mathbb{T}}} \circ S^*_{\Omega_\rho} \circ j^*_\theta$. We note that the group and measure algebra homomorphisms appearing in our factorization are of precisely the same – simply defined and easily studied – canonical types as those appearing in the Cohen factorization. We thus feel that this provides a satisfactory description of the contractive homomorphisms $\varphi : L^1(F) \to M(G)$ which shares the spirit of the Cohen theorem.

As might be expected, our results rest non-trivially upon Greenleaf's beautiful description of the contractive subgroups of $M(G)$ [17]. In Section 4 we complete the topological description of these subgroups and make note of some corollaries. For example, every contractive subgroup with relative weak* topology inherited from $M(G)$ is a topological group with a locally compact completion, also a contractive subgroup of $M(G)$.

In Sections 2 and 3, we establish an explicit, algebraic, one-to-one correspondence between (not necessarily contractive) $*$-homomorphisms, $\varphi : L^1(F) \to M(G)$, and continuous group homomorphisms, $\phi : F \to \mathbb{M}_\phi$, where $\mathbb{M}_\phi$ is a certain semi-topological convolution group in $(M(G), \mathrm{wk}^*)$; in this way the homomorphism problem is, for $*$-homomorphisms, reduced to studying these group homomorphisms. Letting $E(F)$ denote the Eberlein algebra of F – the uniform norm completion of $B(F)$ – the above correspondence is established by showing that any such $*$-homomorphism φ on $L^1(F)$ can be extended to a $w^* - w^*$ continuous $*$-homomorphism, φ_ε, of the involutive dual Banach algebra $E(F)^* (= M(F) \oplus_1 C_0(F)^\perp)$: the map $\varphi_\varepsilon : E(F)^* \to M(G)$ is explicitly described as a dual map $\varphi_\varepsilon = \kappa_\phi^*$, where $\kappa_\phi : C_0(G) \to E(F)$, and $C_0(G)$ is the algebra of continuous functions on G which vanish at infinity. Note that $C_0(G)$ is the uniform norm completion of the Fourier algebra, $A(F)$, and κ_ϕ is algebraically described with respect to a continuous group homomorphism $\phi : F \to \mathbb{M}_\phi$ in terms of its action on coefficient functions of the left regular representation of G. This explicit construction of a convolution algebra homomorphism κ_ϕ^* from a group homomorphism ϕ is employed repeatedly in the latter part of the paper dealing with contractive homomorphisms. Observe that if F is identified, via point evaluations on $E(F)$, with a subset of $E(F)^*$, then its linear span, $\mathbb{C}F$, is wk*-dense in $E(F)^*$. Thus, φ_ε is *topologically* determined by its values on F. However, $E(F)^*$ can be extremely large in comparison with $\mathbb{C}F$ – the norm closure of $\mathbb{C}F$ in $E(F)^*$ is $\ell^1(F)$ – so this, unlike the aforementioned algebraic construction from Section 3, does not provide a very satisfying mechanism for recovering φ_ε, and therefore φ, from its values on F.

In Section 6, we list some consequences of our factorization theorems from Section 5. For example, we completely describe all $w^* - w^*$ continuous contractive homomorphisms $\varphi : \mathfrak{X}(F)^* \to M(G)$ and all $w^* - w^*$ continuous contractive homomorphisms $\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^*$ mapping at least one element μ_0 in $M(F)$ to a non-zero element of $M(G)$; here $\mathfrak{X}$ denotes one of *LUC*, *WAP* or E where $LUC(F)$ and $WAP(F)$ are respectively the algebras of uniformly continuous, and weakly almost periodic, functions on F. The famous theorems of Wendel, Johnson and Strichartz show that the isometric isomorphisms $L^1(F) \to L^1(G)$ and $M(F) \to M(G)$ are always determined by a continuous character $\alpha \in \widehat{F}^1$ and a topological isomorphism $\phi : F \to G$; we extend this by characterizing all homomorphisms between convolution algebras determined by some $\alpha \in \widehat{F}^1$ and a continuous homomorphism $\phi : F \to G$. As well, we characterize all contractive $w^* - w^*$ continuous isomorphisms of $\mathfrak{X}(F)^*$ onto $\mathfrak{X}(G)^*$. A simple characterization of all contractive epimorphisms $\varphi : L^1(F) \to L^1(G)$ is also provided.

1. Preliminary results and notation

Throughout this paper, F and G are locally compact groups with left Haar measures dx. Let $\mathfrak{X}$ be a (closed, linear) subspace of $CB(G)$ (or $\ell^\infty(G)$). Then $\mathfrak{X}$ is called *left* (*right*) *invariant* if $f \cdot x \in \mathfrak{X}$ (resp. $x \cdot f \in \mathfrak{X}$) whenever $f \in \mathfrak{X}$ and $x \in G$. Here, $f \cdot x(y) = l_x f(y) = f(xy)$ and

$x \cdot f(y) = r_x f(y) = f(yx)$ $(y \in G)$. If $\mathfrak{X}$ is left (right) invariant, then it is *left* (*right*) *introverted* in $CB(G)$ if $m \cdot f \in \mathfrak{X}$ $(f \cdot m \in \mathfrak{X})$ whenever $f \in \mathfrak{X}$ and $m \in \mathfrak{X}^*$, where $m \cdot f(x) = m(f \cdot x)$ $(f \cdot m(x) = m(x \cdot f))$. If $\mathfrak{X}$ is both left and right introverted, then it is *introverted.* When $\mathfrak{X}$ is left (right) introverted, $\mathfrak{X}^*$ is a Banach algebra with left (right) Arens product defined by

$$n \square m(f) = n(m \cdot f) \quad \big((n \diamond m)(f) = m(f \cdot n)\big)\; n, m \in \mathfrak{X}^*,\ f \in \mathfrak{X}.$$

If $\mathfrak{X}$ is introverted in $CB(G)$ and left and right Arens product agree on $\mathfrak{X}^*$, then $\mathfrak{X}$ is *Arens regular.* In this paper, we are interested in $LUC(G)$ $(RUC(G))$, the left (right) introverted space of left (right) uniformly continuous functions on G, and $WAP(G)$ the introverted space of weakly almost periodic functions on G. We are especially interested in the introverted space $E(G)$, the uniform norm completion of the Fourier–Stieltjes algebra, $B(G)$, of G; $E(G) \subseteq WAP(G)$ is called the Eberlein algebra of G. For more information about introverted spaces and Arens products, see [3] and [4], for example.

Unless otherwise stated, $\mathfrak{X}(G)$ will always denote a left introverted subspace of $CB(G)$ which contains $C_0(G)$, the continuous functions on G vanishing at infinity. The map

$$\Theta_G : M(G) \to \mathfrak{X}(G)^* \text{ given by } \quad \big\langle \Theta_G(\mu), \phi \big\rangle = \int_G \phi \, d\mu,\ \mu \in M(G),\ \phi \in \mathfrak{X}(G)$$

defines an isometric isomorphism, and identifying $M(G)$ and $\Theta_G(M(G))$, we have the L^1-direct sum decomposition

$$\mathfrak{X}(G)^* = M(G) \oplus C_0(G)^{\perp}$$

[14, Lemma 1.1]. For this reason, we will use the convolution product notation, $m * n$, in place of $m \square n$ for $m, n \in \mathfrak{X}(G)^*$. (In fact one can, for example, identify Arens product on $WAP(G)^*$ and $E(G)^*$ with convolution product on $M(G^w)$ and $M(G^\varepsilon)$, respectively the measure algebras of the weakly almost periodic and Eberlein compactifications, G^w and G^ε, of G [37].) Letting $I_G : C_0 \hookrightarrow \mathfrak{X}(G)$,

$$\mathfrak{R}_G = I_G^* : \mathfrak{X}(G)^* \to M(G) : m \mapsto m|_{C_0(G)}$$

is an epimorphism such that $\mathfrak{R}_G \circ \Theta_G(\mu) = \mu$ $(\mu \in M(G))$. Let so_l and so_r denote the locally convex left and right strict topologies on $\mathfrak{X}(G)^*$ respectively generated by the semi-norms $p_f(m) = \|f * m\|$ and $q_f(m) = \|m * f\|$ $(f \in L^1(G),\ m \in \mathfrak{X}(G)^*)$.

For $x \in G$, δ_x denotes the Dirac measure at x, and $\Delta_G = \{\delta_x\colon\ x \in G\}$, which we often view as a subset of $\mathfrak{X}(G)^*$; we will sometimes identify $\mathbb{C}G$ with the linear span of Δ_G in $\mathfrak{X}(G)^*$. Note that the map $x \mapsto \delta_x$ is a topological isomorphism of G onto its image in both $(\mathfrak{X}(G)^*, \mathrm{wk}^*)$ and $(\mathfrak{X}(G)^*, so_l)$. For $m \in \mathfrak{X}(G)^*$ multiplication $n \mapsto n * m$ is w^*-continuous; $m \in Z_t(\mathfrak{X}(G)^*)$, the topological centre of $\mathfrak{X}(G)$, if $n \mapsto m * n$ is w^*-continuous. It is always true that $M(G) \subseteq Z_t(\mathfrak{X}(G)^*)$; in fact $M(G) = Z_t(LUC(G)^*)$ [28]. On the other hand, if $\mathfrak{X}(G) \subseteq WAP(G)$, then $Z_t(\mathfrak{X}(G)^*) = \mathfrak{X}(G)^*$ [30]. In particular, $C_0(G)$, $E(G)$ and $WAP(G)$ are all Arens regular. The following simple observation will be used repeatedly.

Lemma 1.1. *Let $\mathfrak{X}(F)$ and $\mathfrak{X}(G)$ be left introverted spaces with $C_0(F) \subseteq \mathfrak{X}(F) \subseteq LUC(F)$ and $C_0(G) \subseteq \mathfrak{X}(G) \subseteq LUC(G)$. Then $(\mathbb{C}F)_{\|\cdot\|\leqslant 1}$ is* wk*-*dense in $(\mathfrak{X}(F)^*)_{\|\cdot\|\leqslant 1}$ and both so_l-dense, and so_r-dense, in $M(F)_{\|\cdot\|\leqslant 1}$. Moreover, a $w^* - w^*$ continuous (resp. $so_r - w^*$ continuous) linear map $\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^*$ (resp. $\varphi : M(F) \to \mathfrak{X}(G)^*$) such that $\varphi(\Delta_F) \subseteq Z_t(\mathfrak{X}(G)^*)$ is a homomorphism if and only if for each $x, y \in F$, $\varphi(\delta_x * \delta_y) = \varphi(\delta_x) * \varphi(\delta_y)$.*

Proof. The first statement is a consequence of the w^*-density of $(\mathbb{C}F)_{\|\cdot\|\leqslant 1}$ in $(\ell^\infty(F)^*)_{\|\cdot\|\leqslant 1}$ and the Hahn–Banach extension theorem; by, for example [17, Lemma 1.1.3], $(\mathbb{C}F)_{\|\cdot\|\leqslant 1}$ is both so_l-dense, and so_r-dense, in $M(G)_{\|\cdot\|\leqslant 1}$. For the less-trivial direction of the second statement, let $m, n \in \mathfrak{X}(F)^*$ and take nets (f_i) and (g_j) in $\mathbb{C}F$ converging w^* to m and n in $\mathfrak{X}(F)^*$. As (f_i) and $(\varphi(f_i))$ lie in the respective topological centres of $\mathfrak{X}(F)^*$ and $\mathfrak{X}(G)^*$, $w^* - w^*$ continuity of φ gives

$$\begin{aligned}\varphi(m * n) &= \lim_i \varphi(f_i * n) = \lim_i \lim_j \varphi(f_i * g_j) = \lim_i \lim_j \varphi(f_i) * \varphi(g_j)\\ &= \lim_i \varphi(f_i) * \varphi(n) = \varphi(m) * \varphi(n),\end{aligned}$$

where all limits are taken in the w^*-topology on $\mathfrak{X}(G)^*$. As multiplication is separately $so_r - w^*$ continuous in $M(F)$, we similarly obtain the bracketed part of the second statement. □

If u is a function on G, define $\check{u}(x) = \overline{u(x^{-1})}$ $(x \in G)$, and call $\mathfrak{X}(G)$ $\vee$-invariant if $\check{u} \in \mathfrak{X}$ whenever $u \in \mathfrak{X}$. For $\vee$-invariant $\mathfrak{X}(G) \subseteq WAP(G)$, it is noted in [37] that

$$m^*(u) = \overline{m(\check{u})}, \quad m \in \mathfrak{X}(G)^*,\ u \in \mathfrak{X}(G)$$

defines an involution on $\mathfrak{X}(G)^*$; also see [9]. Recall that with respect to the canonical right dual module action of $L^1(G)$ on $L^\infty(G)$, $LUC(G) = L^\infty(G) \cdot L^1(G) = LUC(G) \cdot L^1(G)$. We may write $\langle \phi, x\rangle_{\mathfrak{X}^* - \mathfrak{X}}$ to stress that ϕ is being regarded as an element of $\mathfrak{X}^*$, $x \in \mathfrak{X}$; abbreviations such as $LUC^* - LUC$ for $LUC(G)^* - LUC(G)$ will often be used. The following lemma will often be utilized.

Lemma 1.2. *Let $\mathfrak{X}(G)$ be a left introverted subspace of $LUC(G)$, with $C_0(G) \subseteq \mathfrak{X}(G)$.*

(a) *Let $m \in LUC(G)^*$, $\psi = \psi_0 \cdot f_0 \in LUC(G)$ where $\psi_0 \in LUC(G)$, $f_0 \in L^1(G)$. Then*

$$\langle m, \psi\rangle_{LUC^* - LUC} = \langle f_0 * m, \psi_0\rangle_{LUC^* - LUC}.$$

(b) *The identity map $LUC(G)^* \to LUC(G)^*$ and the embedding $\Theta_G : M(G) \hookrightarrow \mathfrak{X}(G)^*$ are $so_l - w^*$ continuous.*
(c) *If $\mathfrak{X}(G) \subseteq RUC(G)$, then Θ_G is $so_r - w^*$ continuous.*
(d) *If $(e_i)_i$ is a right approximate identity in $L^1(G)$, then $e_i \to \delta_{e_G}$ w^* in $\mathfrak{X}(G)^*$.*

Proof. Let $m \in LUC(G)^*$ and let $(h_i)_i$ be a net in $L^1(G)$ such that $h_i \to m$ w^*. Then

$$\begin{aligned}\langle m, \psi\rangle_{LUC^* - LUC} &= \lim \langle \psi_0 \cdot f_0, h_i\rangle_{L^\infty - L^1} = \lim \langle \psi_0, f_0 * h_i\rangle_{L^\infty - L^1}\\ &= \lim \langle f_0 * h_i, \psi_0\rangle_{LUC^* - LUC} = \langle f_0 * m, \psi_0\rangle_{LUC^* - LUC},\end{aligned}$$

because $f_0 \in Z_t(LUC(G)^*)$. This establishes statement (a). For (b), let $m_i \to m$ so_l in $LUC(G)^*$, and take $\psi = \psi_0 \cdot f_0 \in LUC(G)$, written as in part (a). Then

$$\langle m_i - m, \psi\rangle = \langle f * m_i - f * m, \psi_0\rangle \to 0;$$

similarly, Θ_G is $so_l - w^*$ continuous. As so_r convergence in $M(G)$ is equivalent to so_l convergence in $M(G^{op})$ and $RUC(G) = LUC(G^{op})$, part (c) is a consequence of part (b). Obviously, if (e_i) is a right bounded approximate identity, then $e_i \to \delta_{e_G}$ so_l so (d) follows from part (b). □

The dual version of the following result is [24, Proposition 3.5]. The proof here is similar but is included for the convenience of readers who are unfamiliar with the dual setting. Here $\iota_{\mathfrak{X}} : \mathfrak{X} \hookrightarrow \mathfrak{X}^{**}$ denotes the canonical embedding.

Proposition 1.3. *Let $\mathfrak{X}(G)$ be a left introverted subspace of $LUC(G)$ which contains 1_G. The following statements are equivalent*:

(i) *G is compact*;
(ii) *$\Theta_G : M(G) \to \mathfrak{X}(G)^*$ is $w^* - w^*$ continuous*;
(iii) *$\Theta_G : M(G) \to \mathfrak{X}(G)^*$ is surjective.*

Proof. If G is compact, then $\mathfrak{X}(G) = C_0(G)$ and $\Theta = \Theta_G$ is the $w^* - w^*$ continuous identity map. Suppose that Θ is $w^* - w^*$ continuous, with predual map $\Theta_* : \mathfrak{X}(G) \to C_0(G)$. Observe that the mapping $\tau : \mathfrak{X}(G) \to M(G)^*$ given by $\langle \tau(\psi), \mu\rangle = \int \psi\, d\mu$ is an isometry such that $\Theta^* \circ \iota_{\mathfrak{X}(G)} = \tau$. Hence for $\psi \in \mathfrak{X}(G)$,

$$\|\Theta_*(\psi)\| = \|\iota_{C_0(G)}(\Theta_*(\psi))\| = \|\Theta^*(\iota_{\mathfrak{X}(G)}(\psi))\| = \|\tau(\psi)\| = \|\psi\|.$$

In particular, Θ_* has closed range and, because Θ is 1–1, it follows that Θ_* is surjective. Hence, Θ is a bijection because Θ_* is so. Assume now that Θ is surjective, and let $m \in \mathfrak{X}(G)^*$ be such that $m|_{C_0(G)} = 0$. Taking $\mu \in M(G)$ such that $\Theta(\mu) = m$, we have $\mu = \Theta(\mu)|_{C_0(G)} = m|_{C_0(G)} = 0$, whence $m = 0$. By the Hahn–Banach extension theorem, $C_0(G) = \mathfrak{X}(G)$; hence $1_G \in C_0(G)$, and we may conclude that G is compact. □

The set (up to unitary equivalence) of all continuous unitary representations $\{\pi, \mathcal{H}_\pi\}$ of G will be denoted by $\Sigma(G)$. If $\xi, \eta \in \mathcal{H}_\pi$, then $\xi *_\pi \eta(s) = \langle \pi(s)\xi | \eta\rangle$ $(s \in G)$ is the associated coefficient function. The set of all coefficient functions of representations of G, $B(G)$, is the Fourier–Stieltjes algebra of G and is identified with the dual of the group C^*-algebra, $C^*(G)$. This duality satisfies

$$\langle \xi *_\pi \eta, f\rangle_{B-C^*(G)} = \int_G \xi *_\pi \eta(s) f(s)\, ds = \langle \pi(f)\xi \,|\, \eta\rangle \quad \bigl(f \in L^1(G) \subseteq C^*(G)\bigr) \tag{1.1}$$

where on the right π is being viewed as a $*$-representation of $L^1(G)$. We define A_π to be the norm-closed linear subspace of $B(G)$ generated by the coefficient functions of π; VN_π is the von Neumann subalgebra of $B(\mathcal{H}_\pi)$ generated by π on G (or $M(G)$). When π is λ_G, the left regular representation of G, $A_\pi = A(G)$, the Fourier algebra of G, and $VN_\pi = VN(G)$, the group von Neumann algebra of G. For further details, the reader is referred to [19,8,1].

There is a 1–1 correspondence between the unitary representations $\{\pi, \mathcal{H}_\pi\} \in \Sigma(G)$, the non-degenerate $*$-representations $\{\pi_L, \mathcal{H}_\pi\}$ of $L^1(G)$, and the $*$-representations $\{\pi_m, \mathcal{H}_\pi\}$ of $M(G)$. Although $\pi_m : M(G) \to A_\pi^* = VN_\pi \subseteq B(\mathcal{H}_\pi)$ is always $so_l - w^*$ and $so_r - w^*$ continuous, it is only $w^* - w^*$ continuous when $A_\pi \subseteq C_0(G)$ [37]. In [37] it was, however, also shown that there exists a 1–1 correspondence between $\{\pi, \mathcal{H}_\pi\} \in \Sigma(G)$ and the $w^* - w^*$ continuous $*$-representations $\pi_\varepsilon : E(G)^* \to A_\pi^* = VN_\pi \subseteq B(\mathcal{H}_\pi)$, and that $E(G)$ is the smallest introverted subspace of $CB(G)$ with this property. Here

$$\langle \pi_\varepsilon(m)\xi \,|\, \eta\rangle_{\mathcal{H}_\pi} = \langle \pi_\varepsilon(m), \xi *_\pi \eta\rangle_{VN_\pi - A_\pi} = \langle m, \xi *_\pi \eta\rangle_{E^* - E}, \tag{1.2}$$

for $m \in E(G)^*$, $\xi, \eta \in \mathcal{H}_\pi$, and

$$\pi_m = \pi_\varepsilon \circ \Theta_G = \pi_\varepsilon|_{M(G)}, \qquad \pi_L = \pi_m|_{L^1(G)}.$$

To avoid trivial cases, all algebra homomorphisms are assumed to be non-zero.

2. Extending homomorphisms

A Banach algebra $\mathfrak{B}$ which is a dual Banach space and has separately w^*-continuous multiplication is called a dual Banach algebra [35]. For example, $M(G)$, $E(G)^*$, and (more generally) $\mathfrak{X}(G)^*$ for any introverted subspace $\mathfrak{X}(G)$ of $WAP(G)$, are dual Banach algebras. The following result is a special case of a theorem due to Volker Runde [36, Theorem 4.10].

Theorem 2.1 *(Runde). Let $\mathfrak{B}$ be a dual Banach algebra and let $\varphi : L^1(F) \to \mathfrak{B}$ be a bounded homomorphism. Then φ has a unique $w^* - w^*$ continuous homomorphic extension $\varphi_w : WAP(F)^* \to \mathfrak{B}$. Moreover, $\|\varphi_w\| = \|\varphi\|$.*

Greenleaf's canonical extension – see [17, Theorem 4.1.1] – $\varphi_m : M(F) \to M(G)$ of a bounded homomorphism $\varphi : L^1(F) \to M(G)$ has served as a fundamental tool in the study of homomorphisms of group algebras; for example see [17,27,26]. By taking $\mathfrak{B} = M(G)$, the following result improves, and gives a different proof of, Greenleaf's theorem.

Proposition 2.2. *Let $\mathfrak{B}$ be a dual Banach algebra and let $\varphi : L^1(F) \to \mathfrak{B}$ be a bounded homomorphism. Then there exists a unique $so_l - w^*$ and $so_r - w^*$ continuous extension $\varphi_m : M(F) \to \mathfrak{B}$. Moreover, $\|\varphi_m\| = \|\varphi\|$, and if $(e_i)_i$ and $(f_j)_j$ are respectively right and left approximate identities for $L^1(F)$, then*

$$\varphi_m(\mu) = w^* - \lim_i \varphi(\mu * e_i) = w^* - \lim_j \varphi(f_j * \mu), \quad \mu \in M(F).$$

Proof. Let $\varphi_w : WAP(F)^* \to \mathfrak{B}$ be the unique $w^* - w^*$ continuous extension of φ. Then $\varphi_m = \varphi_w \circ \Theta_F : M(F) \to \mathfrak{B}$ is $so_l - w^*$ and $so_r - w^*$ continuous by Lemma 1.2. The displayed equation holds because $\mu * e_i \to \mu$ so_l and $f_j * \mu \to \mu$ so_r in $M(F)$. $\square$

In the dual setting, note that any continuous homomorphism $\varphi : A(F) \to B(G)$ is necessarily a $*$-homomorphism. Indeed, for any such homomorphism there is a continuous map

$\alpha : Y \subset G \to F$ such that $\varphi(u)(x) = u(\alpha(x))$ on Y and zero off Y (see the first paragraph of the proof of Proposition 3.4 in [21]). With Proposition 5.5 we will observe that every contractive homomorphism $\varphi : L^1(F) \to M(G)$ is also a $*$-homomorphism.

The proof of the following theorem explicitly shows how to construct a $w^* - w^*$ continuous $*$-homomorphic extension $\varphi_\varepsilon : E(F)^* \to E(G)^*$ of any $*$-homomorphism $\varphi : L^1(F) \to E(G)^*$. It is not a consequence of [36, Theorem 4.10] (Theorem 2.1) and does not seem to follow from the constructions in Section 3. Another proof of the existence of $\varphi_{\varepsilon m}$ and of the second statement in the theorem is given later but, in these cases, the construction found here seems to be of independent interest. Moreover, for $*$-homomorphisms one can again obtain Proposition 2.2 in the case that $\mathfrak{B}$ is $M(G)$ or $E(G)^*$ from Theorem 2.3, thus making most of this paper self-contained.

Theorem 2.3. *Let $\varphi : L^1(F) \to M(G)$ be a continuous $*$-homomorphism. Then φ has unique $w^* - w^*$ continuous extensions to $*$-homomorphisms*

$$\varphi_\varepsilon : E(F)^* \to E(G)^* \quad \text{and} \quad \varphi_{\varepsilon m} : E(F)^* \to M(G).$$

Moreover, $\varphi_\varepsilon = \kappa_\varphi^*$ where $\kappa_\varphi : E(G) \to E(F)$ maps $B(G)$ contractively into $B(F)$ ($\|\varphi_\varepsilon(u)\|_{B(F)} \leqslant \|u\|_{B(G)}$), and $\|\varphi_\varepsilon\| = \|\varphi_{\varepsilon m}\| = \|\varphi\|$. Every continuous $*$-homomorphism $\varphi : L^1(F) \to E(G)^*$ has a unique extension to a $w^* - w^*$ continuous $*$-homomorphism $\varphi_\varepsilon = \kappa_\varphi^* : E(F)^* \to E(G)^*$.

Proof. Let $\varphi : L^1(F) \to M(G)$ be a continuous $*$-homomorphism. The map $\kappa_\varphi : E(G) \to L^\infty(F)$ is defined in accordance with the following commutative diagrams:

$$\begin{tikzcd} L^1(F) \arrow[r, "\varphi"] \arrow[dr, "\Theta_G \circ \varphi"'] & M(G) \arrow[d, hook, "\Theta_G"] \\ & E(G)^* \end{tikzcd} \qquad \begin{tikzcd} E(G)^{**} \arrow[r, "(\Theta_G \circ \varphi)^*"] & L^\infty(F) \\ E(G) \arrow[u, hook, "\wedge"] \arrow[ur, "\kappa_\varphi"'] & \end{tikzcd} \tag{2.1}$$

We *claim* that κ_φ maps $B(G)$ into $B(F)$ and that κ_φ is contractive with respect to Fourier–Stieltjes algebra norms. To see this, let $u = \xi *_\pi \eta$ where $\{\pi, \mathcal{H}_\pi\} \in \Sigma(G)$. We may assume that $\|u\|_{B(G)} = \|\xi\|\|\eta\|$ [8]. Let $\mathcal{H}_\varphi$ denote the Hilbert subspace of $\mathcal{H}_\pi$ generated by $\{\pi_m(\varphi(f))\xi\colon f \in L^1(F),\ \xi \in \mathcal{H}_\pi\}$, where π_m denotes the $*$-representation of $M(G)$ determined by π. Then $\pi_\varphi(f) = \pi_m(\varphi(f))|_{\mathcal{H}_\varphi}$ defines a non-degenerate $*$-representation of $L^1(F)$ on $\mathcal{H}_\varphi$, and therefore corresponds to a continuous unitary representation $\{\pi_\varphi, \mathcal{H}_\varphi\}$ of F. Observe that because $A = \pi_m(\varphi(L^1(F)))$ is self-adjoint in $B(\mathcal{H}_\pi)$,

$$\langle T\xi | \eta\rangle = \langle T P_\varphi \xi | P_\varphi \eta\rangle, \quad T \in A,\ \xi, \eta \in \mathcal{H}_\pi,$$

where P_φ is the orthogonal projection of $\mathcal{H}_\pi$ onto $\mathcal{H}_\varphi$. We will establish the claim by showing that $\kappa_\varphi(u) = \kappa_\varphi(\xi *_\pi \eta) = P_\varphi \xi *_{\pi_\varphi} P_\varphi \eta$. Indeed, given $f \in L^1(F)$,

$$\begin{aligned}\big\langle \kappa_\varphi(u), f\big\rangle_{L^\infty-L^1(F)} &= \big\langle (\Theta_G\circ\varphi)^*(\hat{u}), f\big\rangle_{L^\infty-L^1(F)} = \big\langle \Theta_G\big(\varphi(f)\big), u\big\rangle_{E^*-E(G)} \\ &= \int_G \big\langle \pi(s)\xi \,\big|\, \eta\big\rangle\, d\big(\varphi(f)\big)(s) = \big\langle \pi_m\big(\varphi(f)\big)\xi \,\big|\, \eta\big\rangle \\ &= \big\langle \pi_m\big(\varphi(f)\big)P_\varphi\xi \,\big|\, P_\varphi\eta\big\rangle = \int_F \big\langle \pi_\varphi(s)P_\varphi\xi \,\big|\, P_\varphi\eta\big\rangle f(s)\,ds \\ &= \langle P_\varphi\xi *_{\pi_\varphi} P_\varphi\eta, f\rangle_{L^\infty-L^1(F)}.\end{aligned}$$

Hence, $\kappa_\varphi(u) \in B(F)$ and $\|\kappa_\varphi(u)\|_{B(F)} \leqslant \|P_\varphi\xi\|\|P_\varphi\eta\| \leqslant \|u\|_{B(G)}$.

As $\kappa_\varphi : E(G) \to L^\infty(F)$ is also continuous with respect uniform norms we also obtain $\kappa_\varphi : E(G) \to E(F)$. Let $\varphi_\varepsilon = \kappa_\varphi^*$. If $f \in L^1(F)$ and $u = \xi *_\pi \eta \in B(G)$, then

$$\begin{aligned}\big\langle \varphi_\varepsilon\big(\Theta_F(f)\big), u\big\rangle_{E^*-E(G)} &= \big\langle \Theta_F(f), \kappa_\varphi(u)\big\rangle_{E^*-E(F)} = \big\langle \kappa_\varphi(u), f\big\rangle_{L^\infty-L^1(F)} \\ &= \big\langle \Theta_G\big(\varphi(f)\big), u\big\rangle_{E^*-E(G)};\end{aligned}$$

that is the diagram

$$\begin{array}{ccc} E(F)^* & \xrightarrow{\varphi_\varepsilon} & E(G)^* \\ \Theta_F \big\uparrow & & \big\uparrow \Theta_G \\ L^1(F) & \xrightarrow{\varphi} & M(G) \end{array}$$

commutes. Let $\varphi_{\varepsilon m} = (\kappa_\varphi \circ I_G)^*$ where $I_G : C_0(G) \hookrightarrow E(G)$. Then $\varphi_{\varepsilon m} : E(F)^* \to M(G)$ satisfies $\varphi_{\varepsilon m} \circ \Theta_F = I_G^* \circ \kappa_\varphi^* \circ \Theta_F = \Re_G \circ \Theta_G \circ \varphi = \varphi$ as needed. Thus φ_ε and $\varphi_{\varepsilon m}$ are $w^* - w^*$ continuous extensions of the $*$-homomorphism φ. As Θ_F is a $*$-homomorphism on $L^1(F)$ with w^*-dense range, and multiplication is separately w^*-continuous on $E(G)^*$ and $M(G)$, it follows that φ_ε and $\varphi_{\varepsilon m}$ are the unique $w^* - w^*$ continuous $*$-homomorphic extensions of φ. Observe that $\|\varphi_\varepsilon\| = \|\varphi_{\varepsilon m}\| = \|\varphi\|$. The case when $\varphi : L^1(F) \to E(G)^*$ follows from the same proof by considering only the second diagram in (2.1) and by using $\pi_\varepsilon : E(F)^* \to B(\mathcal{H}_\pi)$ in place of π_m. □

Remark.

1. Let $\varphi : L^1(F) \to M(G)$ be a $*$-homomorphism. It would be interesting to know if φ_ε necessarily maps $M(F)$ into $M(G)$, i.e. if φ_ε necessarily extends φ_m. This does not seem obvious, but in Section 6 we will see that this is true when φ is contractive.
2. Observe that $\Theta_G \circ \varphi \circ \Re_F : E(F)^* \to E(G)^*$ gives a very simple method of constructing a $*$-homomorphic extension of $\varphi : M(F) \to M(G)$. However, this extension is not necessarily $w^* - w^*$ continuous. Indeed, taking $F = G$ non-compact, the unique $w^* - w^*$ extension of the identity mapping, $id_{M(F)}$, on $M(F)$ is the identity mapping, $id_{E(F)^*}$, on $E(F)^*$. By Proposition 1.3, however, $\Theta_F \circ \varphi \circ \Re_F \neq id_{E(F)^*}$ because Θ_F is not surjective.

3. Group homomorphisms versus $*$-homomorphisms

Let ι_ϕ be a self-adjoint, non-zero idempotent element in $E(G)^*$. Let

$$\mathbb{E}_\phi = \{ m \in E(G)^*\colon\ m^* * m = m * m^* = \iota_\phi \text{ and } m * \iota_\phi = m \}$$

with the relative $\sigma(E(G)^*, E(G))$-topology. When $\iota_\phi \in M(G)$, let

$$\mathbb{M}_\phi = \mathbb{E}_\phi \cap M(G),$$

with the relative $\sigma(M(G), C_0(G))$-topology. For us, bounded sets and nets in $\mathbb{E}_\phi$ (or $\mathbb{M}_\phi$) refer to sets and nets that are norm-bounded in $E(G)^*$ (or $M(G)$).

Lemma 3.1. *Let ι_ϕ be a self-adjoint, non-zero idempotent in $E(G)^*$.*

(i) *With convolution product, identity ι_ϕ, and inverses $m^{-1} = m^*$, $\mathbb{E}_\phi$ and $\mathbb{M}_\phi$ are semi-topological groups with continuous inversion.*
(ii) *For a bounded net $(m_i)_i$ in $\mathbb{E}_\phi$ and $m \in \mathbb{E}_\phi$, the following statements are equivalent*:
  (a) *$m_i \to m$ in $\mathbb{E}_\phi$;*
  (b) *$\omega_\varepsilon(m_i) \to \omega_\varepsilon(m)$ w^* in $W^*(G)$ where $\omega = \omega_G$ is the universal representation of G;*
  (c) *for every $\{\pi, \mathcal{H}_\pi\} \in \Sigma(G)$, $\pi_\varepsilon(m_i) \to \pi_\varepsilon(m)$ w^* in $VN_\pi = A_\pi^*$;*
  (d) *for each $\{\pi, \mathcal{H}_\pi\} \in \Sigma(G)$, $\pi_\varepsilon(m_i) \to \pi_\varepsilon(m)$ WOT in $B(\mathcal{H}_\pi)$.*
(iii) *Let $\lambda = \lambda_G$ be the left regular representation of G. For a bounded net $(\mu_i)_i$ in $\mathbb{M}_\phi$ and $\mu \in \mathbb{M}_\phi$, the following statements are equivalent*:
  (a) *$\mu_i \to \mu$ in $\mathbb{M}_\phi$;*
  (b) *$\lambda_m(\mu_i) \to \lambda_m(\mu)$ w^* in $VN(G)$;*
  (c) *$\lambda_m(\mu_i) \to \lambda_m(\mu)$ WOT in $B(L^2(G))$.*

Proof. It is a simple exercise to check that $\mathbb{E}_\phi$ and $\mathbb{M}_\phi$ are groups. The remainder of part (i) follows from the fact that involution in $E(G)^*$ is w^*-continuous, and multiplication in $E(G)^*$ is separately w^*-continuous. Statement (ii) follows from Eq. (1.2), density of $B(G)$ in $E(G)$, and the assumption that (m_i) is bounded in $\mathbb{E}_\phi$. As λ_G is $w^* - w^*$ continuous on $M(G)$, and $A(G)$ is dense in $C_0(G)$, one similarly obtains statement (iii). □

In the next two subsections we will, among other things, establish an explicit 1–1 correspondence between the $*$-homomorphisms $\varphi : L^1(F) \to E(G)^*$ (respectively $\varphi : L^1(F) \to M(G)$) and the continuous norm-bounded homomorphisms $\phi : F \to \mathbb{E}_\phi$ (respectively $\phi : F \to \mathbb{M}_\phi$).

3.1. Bounded $*$-homomorphisms into $E(G)^*$

Let $\iota_\phi \in E(G)^*$ be a self-adjoint, non-zero idempotent in $E(G)^*$, and let $\phi : F \to \mathbb{E}_\phi$ be a continuous homomorphism. On occasion, we will write ϕ_s instead of $\phi(s)$. Let $\{\pi, \mathcal{H}_\pi\} \in \Sigma(G)$. As $\iota_\phi = \phi(e_F)$ is a self-adjoint idempotent in $E(G)^*$, $P_\phi^\pi = \pi_\varepsilon(\iota_\phi)$ is a projection in $B(\mathcal{H}_\pi)$. Let $\mathcal{H}_\phi^\pi = P_\phi^\pi(\mathcal{H}_\pi)$ and define

$$\pi_\phi(s) = \pi_\varepsilon\big(\phi(s)\big)\big|_{\mathcal{H}_\phi^\pi} \quad (s \in F).$$

When the representation $\{\pi, \mathcal{H}_\pi\}$ is understood to be fixed, we will usually write P_ϕ and $\mathcal{H}_\phi$ instead of P_ϕ^π and $\mathcal{H}_\phi^\pi$.

Lemma 3.2. *If $\{\pi, \mathcal{H}_\pi\} \in \Sigma(G)$, then $\{\pi_\phi, \mathcal{H}_\phi\} \in \Sigma(F)$.*

Proof. First note that for $m \in \mathbb{E}_\phi$, $\pi_\varepsilon(m) = \pi_\varepsilon(\iota_\phi * m) = P_\phi \pi_\varepsilon(m)$, so $\pi_\varepsilon(m)$ maps $\mathcal{H}_\pi$ into $\mathcal{H}_\phi$. For $s \in F$, we therefore have $\pi_\phi(s) \in B(\mathcal{H}_\phi)$ as required. Clearly $\pi_\phi(e_F) = P_\phi|_{\mathcal{H}_\phi} = id_{\mathcal{H}_\phi}$ and for $\xi, \eta \in \mathcal{H}_\phi$ and $s \in F$,

$$\begin{aligned}\left\langle \pi_\phi(s)\xi \,\middle|\, \pi_\phi(s)\eta \right\rangle_{\mathcal{H}_\phi} &= \left\langle \pi_\varepsilon\big(\phi(s)\big)\xi \,\middle|\, \pi_\varepsilon\big(\phi(s)\big)\eta \right\rangle_{\mathcal{H}_\pi} = \left\langle \pi_\varepsilon\big(\phi(s)^*\phi(s)\big)\xi \,\middle|\, \eta \right\rangle_{\mathcal{H}_\pi} \\ &= \langle P_\phi \xi | \eta \rangle_{\mathcal{H}_\pi} = \langle \xi | \eta \rangle_{\mathcal{H}_\phi}.\end{aligned}$$

Moreover, $\pi_\phi(st) = \pi_\phi(s)\pi_\phi(t)$ so π_ϕ is a homomorphism into the group of unitary operators on $\mathcal{H}_\phi$. As $\phi : F \to \mathbb{E}_\phi$ is continuous, it follows from (a) implies (d) of Lemma 3.1(ii) (where the boundedness condition is not required) that $\{\pi_\phi, \mathcal{H}_\phi\}$ is a continuous unitary representation of F. $\square$

We can now define

$$\kappa_\phi : B(G) \to B(F) : \xi *_\pi \eta \mapsto P_\phi \xi *_{\pi_\phi} P_\phi \eta.$$

Lemma 3.3. *The map κ_ϕ is well defined, linear, and if $u = \xi *_\pi \eta \in B(G)$, then*

$$\kappa_\phi u(s) = \left\langle \pi_\varepsilon(\phi_s)\xi \,\middle|\, \eta \right\rangle_{\mathcal{H}_\pi} = \langle \phi_s, u \rangle_{E^*-E(G)} \quad (s \in F).$$

Moreover, $\|\kappa_\phi u\|_{B(F)} \leqslant \|u\|_{B(G)}$ and if $\|\phi_s\| \leqslant L$ $(s \in F)$, then $\|\kappa_\phi u\|_\infty \leqslant L\|u\|_\infty$.

Proof. Observe that

$$\begin{aligned}(P_\phi \xi *_{\pi_\phi} P_\phi \eta)(s) &= \left\langle \pi_\phi(s) P_\phi \xi \,\middle|\, P_\phi \eta \right\rangle_{\mathcal{H}_\phi} = \left\langle \pi_\varepsilon\big(\phi(s)\big) P_\phi \xi \,\middle|\, P_\phi \eta \right\rangle_{\mathcal{H}_\pi} \\ &= \left\langle P_\phi \pi_\varepsilon\big(\phi(s)\big) P_\phi \xi \,\middle|\, \eta \right\rangle_{\mathcal{H}_\pi} = \left\langle \pi_\varepsilon\big(\phi(e_F s e_F)\big)\xi \,\middle|\, \eta \right\rangle_{\mathcal{H}_\pi} \\ &= \left\langle \pi_\varepsilon\big(\phi(s)\big)\xi \,\middle|\, \eta \right\rangle_{\mathcal{H}_\pi} = \langle \phi_s, u \rangle_{E^*-E(G)}\end{aligned}$$

where the last line follows from Eq. (1.2). Thus $\kappa_\phi u(s) = \langle \phi_s, u \rangle_{E^*-E(G)}$, and consequently $\kappa_\phi u$ is independent of the representation of u as a coefficient function, κ_ϕ is linear, and $\|\kappa_\phi u\|_\infty \leqslant L\|u\|_\infty$ when $\|\phi_s\| \leqslant L$ $(s \in F)$. Assuming without loss of generality that $u = \xi *_\pi \eta$ with $\|u\|_{B(G)} = \|\xi\|\|\eta\|$ [8], we obtain $\|\kappa_\phi u\|_{B(F)} = \|P_\phi \xi *_{\pi_\phi} P_\phi \eta\|_{B(F)} \leqslant \|P_\phi \xi\|\|P_\phi \eta\| \leqslant \|\xi\|\|\eta\| = \|u\|_{B(G)}$. $\square$

Assume now that the continuous homomorphism $\phi : F \to \mathbb{E}_\phi$ satisfies $\|\phi_s\| \leqslant L$ $(s \in F)$. By Lemma 3.3, $\kappa_\phi : B(G) \to B(F)$ extends to a linear map $\kappa_\phi : E(G) \to E(F)$ such that $\|\kappa_\phi\| \leqslant L$. It is easy to see that we again have the formula

$$\kappa_\phi u(s) = \langle \phi_s, u \rangle_{E^*-E(G)} \quad \big(u \in E(G),\ s \in F\big).$$

Theorem 3.4. *Let $\phi : F \to \mathbb{E}_\phi$ be a continuous homomorphism such that $\sup_{s\in F} \|\phi(s)\| = L$. Then*

$$\kappa_\phi^* : E(F)^* \to E(G)^*$$

is a $w^* - w^*$ continuous $*$-homomorphism on $E(F)^*$ with $\|\kappa_\phi^*\| = L$. Moreover, κ_ϕ^* is unique among all such $w^* - w^*$ continuous linear maps satisfying $\kappa_\phi^*(\delta_s) = \phi(s)$ $(s \in F)$.

Proof. For $s \in F$ and $u \in E(G)$, $\langle \kappa_\phi^*(\delta_s), u\rangle = \kappa_\phi u(s) = \langle \phi_s, u\rangle$ so $\kappa_\phi^*(\delta_s) = \phi(s)$ as claimed. As $\phi : F \to \mathbb{E}_\phi$ is a homomorphism such that $\phi(s^{-1}) = \phi(s)^*$, it follows from Lemma 1.1 that κ_ϕ^* is a $*$-homomorphism on $E(F)^*$. In accordance with the objectives stated in the introduction to this paper, we now give a second, algebraic proof of this fact: Let $m, n \in E(F)^*$, $u \in E(G)$, $s, t \in F$. Then

$$\kappa_\phi(u \cdot \phi_s)(t) = \langle \phi_t, u \cdot \phi_s\rangle = \langle \phi_s * \phi_t, u\rangle = \langle \phi_{st}, u\rangle = \kappa_\phi u(st) = (\kappa_\phi u) \cdot s(t),$$

so

$$\begin{aligned}\kappa_\phi\big(\kappa_\phi^*(n) \cdot u\big)(s) &= \big\langle \phi_s, \kappa_\phi^*(n) \cdot u\big\rangle = \big\langle \phi_s * \kappa_\phi^*(n), u\big\rangle = \big\langle n, \kappa_\phi(u \cdot \phi_s)\big\rangle \\ &= \big\langle n, (\kappa_\phi u) \cdot s\big\rangle = n \cdot (\kappa_\phi u)(s).\end{aligned}$$

Hence,

$$\big\langle \kappa_\phi^*(m * n), u\big\rangle = \big\langle m, n \cdot (\kappa_\phi u)\big\rangle = \big\langle m, \kappa_\phi\big(\kappa_\phi^*(n) \cdot u\big)\big\rangle = \big\langle \kappa_\phi^*(m) * \kappa_\phi^*(n), u\big\rangle,$$

as needed. Also,

$$(\kappa_\phi u)^\vee(s) = \overline{\kappa_\phi u\big(s^{-1}\big)} = \overline{\langle \phi_{s^{-1}}, u\rangle} = \overline{\big\langle \phi_s^*, u\big\rangle} = \langle \phi_s, \check{u}\rangle = \kappa_\phi(\check{u})(s),$$

from which it follows that $\langle \kappa_\phi^*(m^*), u\rangle = \langle \kappa_\phi^*(m)^*, u\rangle$. $\square$

We now explicitly describe the 1–1 correspondence between bounded $*$-homomorphisms into $E(G)^*$ and continuous group homomorphisms into $\mathbb{E}_\phi$: Let

$$\varphi : L^1(F) \to E(G)^*, \qquad \varphi_m : M(F) \to E(G)^* \quad \text{and} \quad \varphi_\varepsilon = \kappa_\varphi^* : E(F)^* \to E(G)^*$$

be a continuous $*$-homomorphism, and its respective $so_l/so_r - w^*$ and $w^* - w^*$ continuous extensions to $M(F)$ and $E(F)^*$; see Proposition 2.2 and Theorem 2.3. Letting $\iota_\phi = \varphi_m(\delta_{e_F})$,

$$\phi : F \to \mathbb{E}_\phi \text{ defined by} \quad \phi(s) = \varphi_m(\delta_s) \ (s \in F)$$

is a continuous homomorphism. We can hence form

$$\kappa_\phi : E(G) \to E(F) \quad \text{and} \quad \kappa_\phi^* : E(F)^* \to E(G)^*$$

as in Theorem 3.4. For $x \in F$, $\kappa_\phi^*(\delta_x) = \phi(x) = \varphi_m(\delta_x)$, so $\kappa_\phi^* \circ \Theta_F = \varphi_m$ by Lemma 1.1. Hence, κ_ϕ^* extends φ so, by Theorem 2.3, $\kappa_\phi^* = \varphi_\varepsilon = \kappa_\varphi^*$ and $\kappa_\phi = \kappa_\varphi$ (observe that this gives another proof of the last statement in Theorem 2.3). Let (f_i) be an approximate identity for $L^1(F)$. In terms of φ,

$$\phi(s) = w^* - \lim_i \varphi(\delta_s * f_i) = w^* - \lim_i \varphi(f_i * \delta_s) = \phi_\varphi(s) \quad (s \in F) \tag{3.1}$$

by Proposition 2.2, so

$$\begin{aligned}\kappa_\phi u(s) &= \langle \phi(s), u \rangle_{E^*-E(G)} = \lim_i \langle \varphi(\delta_s * f_i), u \rangle_{E^*-E(G)} \\ &= \kappa_\varphi u(s) \quad \big(u \in E(G),\ s \in F\big).\end{aligned} \tag{3.2}$$

In summary, we have established the following theorem.

Theorem 3.5. *There exists a* 1–1 *correspondence between*:

- *bounded $*$-homomorphisms $\varphi : L^1(F) \to E(G)^*$;*
- *$so_l/so_r - w^*$ continuous bounded $*$-homomorphisms $\varphi_m : M(F) \to E(G)^*$;*
- *$w^* - w^*$ continuous $*$-homomorphisms $\varphi_\varepsilon : E(F)^* \to E(G)^*$; and*
- *continuous, bounded homomorphisms $\phi : F \to \mathbb{E}_\phi$.*

The correspondence is given by

$$\phi = \varphi_\varepsilon|_{\Delta_F} = \varphi_m|_{\Delta_F} = \phi_\varphi \leftrightarrow \varphi_\varepsilon = \kappa_\phi^* = \kappa_\varphi^* \leftrightarrow \varphi = \varphi_\varepsilon|_{L^1(F)} = \varphi_m|_{L^1(F)} = \kappa_\phi^*\big|_{L^1(F)}.$$

3.2. Bounded $*$-homomorphisms into $M(G)$

In [17], Greenleaf established a forward link from the set of contractive homomorphisms $\varphi : L^1(F) \to M(G)$ into the set of continuous homomorphisms mapping F into the contractive subgroups of $M(G)$. In this subsection we establish an explicit 1–1 correspondence between the set of all bounded $*$-homomorphisms $\varphi : L^1(F) \to M(G)$ and the set of continuous bounded homomorphisms $\phi : F \to \mathbb{M}_\phi$.

Let $\phi : F \to \mathbb{M}_\phi$ be a continuous homomorphism, and let $\lambda = \lambda_G$ denote the left regular representation of G on $L^2(G)$. Define $P_\phi = P_\phi^\lambda$ and $\{\lambda_\phi, \mathcal{H}_\phi^\lambda\}$ as in Section 3.1. Observe that because $\lambda_m = \lambda_\varepsilon|_{M(G)}$ and $\mathbb{M}_\phi \subseteq M(G)$, λ_ε can be replaced by λ_m in the definition of λ_ϕ. Applying Lemma 3.1(iii) we now obtain that $\{\lambda_\phi, \mathcal{H}_\phi^\lambda\}$ is a continuous unitary representation of F. Define

$$\kappa_\phi : A(G) \to B(F) : \xi *_\lambda \eta \mapsto P_\phi \xi *_{\lambda_\phi} P_\phi \eta.$$

The proof of Lemma 3.3 also applies to give the following lemma. Recall that $\lambda_m(\mu)\xi = \mu * \xi$ $(\mu \in M(G),\ \xi \in L^2(G))$.

Lemma 3.6. *The map κ_ϕ is well defined, linear, and for each $u = \xi *_\lambda \eta \in A(G)$,*

$$\kappa_\phi u(s) = \big\langle \lambda_m\big(\phi(s)\big)\xi \big| \eta \big\rangle_{L^2(G)} = \langle \phi_s * \xi | \eta \rangle_{L^2(G)} = \int_G u \, d\phi_s \quad (s \in F).$$

Moreover, $\|\kappa_\phi u\|_{B(F)} \leqslant \|u\|_{A(G)}$ and if $\|\phi_s\| \leqslant L$ $(s \in F)$, then $\|\kappa_\phi u\|_\infty \leqslant L\|u\|_\infty$.

Hence, if $\phi : F \to \mathbb{M}_\phi$ is bounded by L, then $\kappa_\phi : A(G) \to B(F)$ extends to a bounded linear map $\kappa_\phi : C_0(G) \to E(F)$ such that $\|\kappa_\phi\| \leqslant L$, and the formula

$$\kappa_\phi u(s) = \int_G u \, d\phi_s \quad \big(u \in C_0(G),\ s \in F\big)$$

holds. As in Section 3.1 we obtain:

Theorem 3.7. *The map $\kappa_\phi^* : E(F)^* \to M(G)$ is a $w^* - w^*$ continuous $*$-homomorphism with $\|\kappa_\phi^*\| = \sup\{\|\phi_s\|:\ s \in F\}$. Moreover, κ_ϕ^* is unique among all such $w^* - w^*$ continuous linear maps satisfying $\kappa_\phi^*(\delta_s) = \phi(s)$ $(s \in F)$.*

Let

$$\varphi : L^1(F) \to M(G), \qquad \varphi_m : M(F) \to M(G) \quad \text{and} \quad \varphi_{\varepsilon m} : E(F)^* \to M(G)$$

be a continuous $*$-homomorphism, and its respective $so_l/so_r - w^*$ and $w^* - w^*$ continuous extensions to $M(F)$ and $E(F)^*$; see Proposition 2.2 and Theorem 2.3. Letting $\iota_\phi = \varphi_m(\delta_{e_F})$,

$$\phi : F \to \mathbb{M}_\phi \text{ defined by } \quad \phi(s) = \varphi_m(\delta_s)\ (s \in F)$$

is a continuous homomorphism. As in Section 3.1, Theorem 3.7 implies that κ_ϕ^* is a $w^* - w^*$ continuous $*$-homomorphism which extends φ_m and therefore φ; hence $\kappa_\phi^* = \varphi_{\varepsilon m}$. Observe as well that Eqs. (3.1) and (3.2) hold with the $E(G)^* - E(G)$ pairing replaced by $M(G) - C_0(G)$. We have established the following result.

Theorem 3.8. *There exists a* 1–1 *correspondence between*:

- *bounded $*$-homomorphisms $\varphi : L^1(F) \to M(G)$;*
- *$so_l/so_r - w^*$ continuous bounded $*$-homomorphisms $\varphi_m : M(F) \to M(G)$;*
- *$w^* - w^*$ continuous $*$-homomorphisms $\varphi_{\varepsilon m} : E(F)^* \to M(G)$; and*
- *continuous, bounded homomorphisms $\phi : F \to \mathbb{M}_\phi$.*

The correspondence is given by

$$\phi = \varphi_{\varepsilon m}|_{\Delta_F} = \varphi_m|_{\Delta_F} = \phi_\varphi \leftrightarrow \varphi_{\varepsilon m} = \kappa_\phi^* \leftrightarrow \varphi = \varphi_{\varepsilon m}|_{L^1(F)} = \varphi_m|_{L^1(F)} = \kappa_\phi^*\big|_{L^1(F)}.$$

Observe that it follows from Lemmas 3.3 and 3.6, and Theorems 3.5 and 3.8, that bounded $*$-homomorphisms $\varphi : L^1(F) \to E(G)^*$ and $\varphi : L^1(F) \to M(G)$ determine $*$-homomorphisms $\varphi : W^*(F) \to W^*(G)$ and $\varphi : W^*(F) \to VN(G)$ respectively; here $W^*(G) = C^*(G)^{**}$. The author intends to study $*$-homomorphisms of group von Neumann algebras and group C^*-algebras from this perspective in future work.

4. Contractive subgroups of $M(G)$

In this section we will augment Greenleaf's characterization of the contractive subgroups of $M(G)$ [17] by completing the description of their topological structures. We begin by recalling some of Greenleaf's work and fixing some notation that will be used throughout the remainder of this paper.

Throughout this section, Γ denotes a fixed non-zero subgroup of the unit ball of $M(G)$ with identity ι_Γ and relative wk*-*topology from $M(G)$.* There is a compact subgroup K of G and $\rho \in \widehat{K}^1$ – that is, ρ is a continuous homomorphism of K into the circle group $\mathbb{T}$ – such that $\iota_\Gamma = \rho m_K$. Here, m_K denotes normalized Haar measure on K and ρm_K is viewed as an element of $M(G)$ via $\langle \rho m_K, f\rangle = \int_K f\rho\, dm_K$ $(f \in C_0(G))$. Letting

$$H_0 = \bigcup_{\mu\in\Gamma} \operatorname{supp}\mu \quad \text{and} \quad K_0 = \ker\rho,$$

H_0 is a subgroup of G, K and K_0 are compact normal subgroups of H_0, and K/K_0 is contained in the centre of H_0/K_0. Moreover, if

$$\Omega = \bigl\{(\alpha,t) \in \mathbb{T}\times H\colon\ \alpha\delta_t * \rho m_K \in \Gamma\bigr\},$$

then Ω is a subgroup of $\mathbb{T}\times G$ with $p_G(\Omega) = H_0$ and

$$\Gamma = \Gamma_\Omega = \bigl\{\alpha\delta_t * \rho m_K\colon\ (\alpha,t)\in\Omega\bigr\};$$

here $p_G : \mathbb{T}\times G \to G$ is the projection onto G. Observe that if we introduce the notation $\Omega_\Gamma = \Omega$, then this says that $\Gamma_{\Omega_\Gamma} = \Gamma$. The statements in this paragraph are all contained in [17, Theorem 3.1.8].

We let $H = \overline{H_0}$ and refer to H as the *support subgroup* of Γ. (Note that unless H_0 is closed in H, in [17] and [27] H denotes a different locally compact group.) It is convenient to observe that, by the Tietze extension theorem, a net (μ_i) converges to μ in Γ if and only if $\mu_i \to \mu$ wk* in $M(H)$. We let

$$\Omega_\rho = \bigl\{\bigl(\rho(k),k\bigr)\colon\ k\in K\bigr\}$$

(denoted Ω_0 in [17]), and let $\overline{\Omega}$ denote the closure of Ω in $\mathbb{T}\times G$. The following lemma records some consequences of [17] that we will find useful.

Lemma 4.1. *Let $\Gamma_{\overline{\Omega}} = \{\alpha\delta_t * \rho m_K\colon\ (\alpha,t)\in\overline{\Omega}\}$ and $\Gamma_{\mathbb{T}\times H} = \{\alpha\delta_t * \rho m_K\colon\ (\alpha,t)\in\mathbb{T}\times H\}$. Then*:

- (i) *$\iota_\Gamma = \rho m_K$ is a self-adjoint idempotent in $M(G)$. In the notation Section* 3, *Γ is a subgroup of $\mathbb{M}_\phi$ when $\iota_\phi = \rho m_K$.*
- (ii) *$\Gamma_{\overline{\Omega}}$ and $\Gamma_{\mathbb{T}\times H}$ are contractive subgroups of $M(G)$, each with support subgroup H and identity ρm_K. Moreover, $\Omega_{\Gamma_{\overline{\Omega}}} = \overline{\Omega}$ and $\Omega_{\Gamma_{\mathbb{T}\times H}} = \mathbb{T}\times H$.*
- (iii) *$\Gamma_{\mathbb{T}\times H}$ is the largest contractive subgroup of $M(G)$ with support subgroup contained in H and identity ρm_K.*

(iv) *The map* $\phi_0(\alpha, t) = \alpha\delta_t * \rho m_K$ *defines a continuous epimorphism of* $\Omega = \Omega_\Gamma$ *onto* Γ, $\overline{\Omega}$ *onto* $\Gamma_{\overline{\Omega}}$, *and* $\mathbb{T} \times H$ *onto* $\Gamma_{\mathbb{T}\times H}$, *with kernel* Ω_ρ.

Proof. (i) For $u \in C_0(G)$,

$$\langle (\rho m_K)^*, u \rangle = \overline{\int_H \overline{u(s^{-1})}\rho(s)\,dm_K(s)} = \int_K u(s^{-1})\rho(s^{-1})\,dm_K(s)$$
$$= \int_K u(s)\rho(s)\,dm_K(s) = \langle \rho m_K, u \rangle,$$

so $\iota_\phi = \rho m_K$ is a self-adjoint idempotent in $M(G)$. Moreover, if $\mu = \alpha\delta_t * \rho m_K \in \Gamma = \Gamma_\Omega$, then $\mu^* = \overline{\alpha}\delta_{t^{-1}} * \rho m_K \in \Gamma_\Omega$ and $\mu * \mu^* = \mu^* * \mu = \iota_\phi$. Hence, Γ is a subgroup of $\mathbb{M}_\phi$.

The remaining statements follow from Theorem 3.1.8 and Lemma 3.1.11 of [17]:

(ii) As $H = \overline{H_0}$, it is clear from the above discussion that K and K_0 are compact normal subgroups of H with K/K_0 in the centre of H/K_0. Obviously, $p_G(\mathbb{T} \times H) = H$ and because $p_G(\Omega) = H_0$, continuity of p_G and compactness of $\mathbb{T}$ give $p_G(\overline{\Omega}) = H$. Theorem 3.1.8 of [17] hence tells us that $\Gamma_{\overline{\Omega}}$ and $\Gamma_{\mathbb{T}\times H}$ are contractive subgroups of $M(G)$ with identity ρm_K and $H = \bigcup_{\mu\in\Gamma_{\overline{\Omega}}} \operatorname{supp}(\mu) = \bigcup_{\mu\in\Gamma_{\mathbb{T}\times H}} \operatorname{supp}(\mu)$. Also by [17, Theorem 3.1.8], $\overline{\Omega} \subseteq \Omega_{\Gamma_{\overline{\Omega}}} = \overline{\Omega} \cdot \Omega_\rho = \overline{\Omega}$ because $\Omega_\rho \subseteq \Omega \subseteq \overline{\Omega}$. Hence, $\Omega_{\Gamma_{\overline{\Omega}}} = \overline{\Omega}$ and similarly, $\Omega_{\Gamma_{\mathbb{T}\times H}} = \mathbb{T} \times H$.

(iii) If Γ' is any contractive subgroup of $M(G)$ with support subgroup H' contained in H and identity ρm_K, then by [17, Theorem 3.1.8], $\Gamma' = \Gamma_{\Omega_{\Gamma'}}$ where $\Omega_{\Gamma'}$ is a subgroup of $\mathbb{T} \times G$ and $p_G(\Omega_{\Gamma'}) \subseteq H' \subseteq H$. Therefore, $\Omega_{\Gamma'} \subseteq \mathbb{T} \times H$ and hence $\Gamma' = \Gamma_{\Omega_{\Gamma'}} \subseteq \Gamma_{\mathbb{T}\times H}$.

(iv) This is an immediate consequence of part (ii) and [17, Lemma 3.1.11]. □

By Lemma 4.1, the map $\phi((\alpha, t)\Omega_\rho) = \alpha\delta_t * \rho m_K$ defines a continuous group isomorphism of Ω/Ω_ρ onto Γ, $\overline{\Omega}/\Omega_\rho$ onto $\Gamma_{\overline{\Omega}}$, and $\mathbb{T} \times H/\Omega_\rho$ onto $\Gamma_{\mathbb{T}\times H}$. We now prove that in each case ϕ is actually a topological isomorphism.

Theorem 4.2. *For any contractive subgroup* Γ *of* $M(G)$, *the map*

$$\phi : \Omega/\Omega_\rho \to \Gamma : (\alpha, t)\Omega_\rho \mapsto \alpha\delta_t * \rho m_K$$

is a topological isomorphism.

Proof. We have already observed that ϕ is a continuous group isomorphism. We know that multiplication in Γ is separately continuous, so it suffices to establish continuity of the group isomorphism

$$\phi^{-1} : \Gamma \to \Omega/\Omega_\rho : \alpha\delta_t * \rho m_K \mapsto (\alpha, t)\Omega_\rho$$

at $\iota_\Gamma = \rho m_K$. To this end, we begin with a net $((\alpha_i, t_i))_i$ in Ω such that

$$\mu_i = \alpha_i\delta_{t_i} * \rho m_K \to \rho m_K \quad \text{wk}^* \text{ in } M(H) \tag{4.1}$$

and will show that $\lim_j \phi^{-1}(\mu_{i_j}) = e_{\Omega/\Omega_\rho}$ in Ω/Ω_ρ for some subnet $(\mu_{i_j})_j$ of $(\mu_i)_i$.

We *claim* first that $\lim_i q_K(t_i) = q_K(e_H)$ where $q_K : H \to H/K$ is the canonical homomorphism. If this is not the case, then we can choose a relatively compact open neighbourhood U of $e_{H/K}$ such that, by passing to a subnet if necessary, for each i, $q_K(t_i) \notin U$. Taking U_K to be a relatively compact neighbourhood of K in H such that $q_K(U_K) = U$, for each i and each $k \in K$, $t_i k \notin U_K$. Choosing $v \in C_0(H)$ such that v vanishes off U_K and $v|_K = \bar{\rho}$, (4.1) gives

$$0 = \alpha_i \int_K v(t_i k)\rho(k)\,dm_K(k) = \langle \mu_i, v\rangle \to \langle \iota_\Gamma, v\rangle = \int_K v(k)\rho(k)\,dm_K(k) = 1.$$

This contradiction establishes our claim.

It follows that we can find a relatively compact neighbourhood W of K such that (t_i) is eventually in W: letting W_0 be any relatively compact neighbourhood of K, let $W = W_0 K$. Let $(t_{i_j})_j$ be a subnet of $(t_i)_i$ and $k_0 \in \overline{W}$ such that $\lim_j t_{i_j} = k_0$. Then $e_H K = \lim_j t_{i_j} K = k_0 K$ and hence $\lim_j t_{i_j} = k_0 \in K$. Therefore $\delta_{t_{i_j}} \to \delta_{k_0}$ wk* in $M(H)$ and it follows that

$$\rho(k_0)\delta_{t_{i_j}} * \rho m_K \to \rho(k_0)\delta_{k_0} * \rho m_K = \rho m_K \quad \text{wk* in } M(H). \tag{4.2}$$

Taking $v \in C_0(H)$ such that $\langle \rho m_K, v\rangle = \int_K v(k)\rho(k)\,dm_K(k) = 1$, (4.1) and (4.2) give

$$\alpha_{i_j}\langle \delta_{t_{i_j}} * \rho m_K, v\rangle \to \langle \rho m_K, v\rangle = 1 \quad \text{and} \quad \rho(k_0)\langle \delta_{t_{i_j}} * \rho m_K, v\rangle \to \langle \rho m_K, v\rangle = 1.$$

Letting $\gamma_{i_j} = \langle \delta_{t_{i_j}} * \rho m_K, v\rangle$, these limits may be written as

$$\alpha_{i_j}\gamma_{i_j} \to 1 \quad \text{and} \quad \gamma_{i_j} \to \overline{\rho(k_0)}.$$

Hence,

$$\left|\alpha_{i_j} - \rho(k_0)\right| = \frac{|\gamma_{i_j}\alpha_{i_j} - \gamma_{i_j}\rho(k_0)|}{|\gamma_{i_j}|} \to \frac{|1 - \overline{\rho(k_0)}\rho(k_0)|}{|\overline{\rho(k_0)}|} = 0.$$

Thus, $(\alpha_{i_j}, t_{i_j}) \to (\rho(k_0), k_0) \in \Omega_\rho$ in Ω giving

$$\phi^{-1}(\mu_{i_j}) = (\alpha_{i_j}, t_{i_j})\Omega_\rho \to \big(\rho(k_0), k_0\big)\Omega_\rho = e_{\Omega/\Omega_\rho}$$

as needed. □

Corollary 4.3. *Let Γ be a contractive subgroup of $M(G)$. Then Γ is a topological group and Γ has completion $\Gamma_{\overline{\Omega}}$ which is a locally compact contractive subgroup of $M(G)$. Moreover, $\Gamma_{\mathbb{T}\times H}$ is also a locally compact contractive subgroup of $M(G)$ and the map*

$$\phi : (\alpha, t)\Omega_\rho \mapsto \alpha\delta_t * \rho m_K$$

is a topological isomorphism yielding

$$\Gamma \cong \Omega/\Omega_\rho, \qquad \Gamma_{\overline{\Omega}} \cong \overline{\Omega}/\Omega_\rho, \quad \textit{and} \quad \Gamma_{\mathbb{T}\times H} \cong \mathbb{T} \times H/\Omega_\rho.$$

Proof. By Theorem 4.2, $\Gamma \cong \Omega/\Omega_\rho$ is a topological group. Also, by Theorem 4.2 applied to $\Gamma_{\overline{\Omega}}$ and $\Omega_{\Gamma_{\overline{\Omega}}} = \overline{\Omega}$ – see Lemma 4.1 – ϕ is a topological isomorphism giving $\Gamma_{\overline{\Omega}} \cong \overline{\Omega}/\Omega_\rho$. Hence, $\Gamma_{\overline{\Omega}}$ is a locally compact group. Similarly, $\Gamma_{\mathbb{T}\times H}$ is a locally compact group topologically isomorphic, via ϕ, to $\mathbb{T} \times H/\Omega_\rho$. As Ω/Ω_ρ has closure equal to the locally compact group $\overline{\Omega}/\Omega_\rho$ in $\mathbb{T} \times H/\Omega_\rho$, and $\phi : \overline{\Omega}/\Omega_\rho \to \Gamma_{\overline{\Omega}}$ maps Ω/Ω_ρ onto Γ, Γ has locally compact completion $\Gamma_{\overline{\Omega}}$. $\square$

5. Contractive homomorphisms of group algebras

As we have already mentioned in the introduction, H.L. Pham recently extended the contractive version of Cohen's theorem by showing that every contractive homomorphism $\varphi : A(F) \to B(G)$ factors as $\varphi = l_{r_0} \circ s \circ j_\theta \circ l_{u_0}$ [33, Theorem 5.1]; see the diagram (0.1). Assuming that F and G are abelian, one can check that the precise dual form of the homomorphisms l_r, j_ϕ and s are the maps $A_\alpha = M_\alpha^*$, j_ϕ^* and S_K^* defined below. Thus, when F and G are abelian, every contractive homomorphism $\varphi : L^1(F) \to M(G)$ factors as

$$\varphi = A_{\rho_H} \circ S_K^* \circ j_{\theta_K}^* \circ A_\alpha : L^1(F) \to M(H) \hookrightarrow M(G), \tag{5.1}$$

where $\rho_H \in \widehat{H}^1$, $\alpha \in \widehat{F}^1$, K is a compact normal subgroup of a closed subgroup H of G and $\theta_K : F \to H/K$ is a continuous homomorphism. For this reason we will refer to a factorization (5.1) of a contractive homomorphism $\varphi : L^1(F) \to M(G)$ as a *Cohen factorization*; in the abelian case we can use G instead of H. On p. 449 of [27], the authors naturally raised the question of whether a contractive homomorphism $\varphi : L^1(F) \to M(G)$ always has a Cohen factorization, or a factorization of the form described in [27, Theorem 1(i)]. In this section we will answer this question (negatively) with Example 5.4. We will also prove Theorem 5.11 which provides an alternative factorization of any contractive homomorphism into four, canonically defined, homomorphisms.

5.1. Canonical homomorphisms and Cohen factorizations

We begin this subsection by defining the canonical maps $A_\alpha = M_\alpha^*$, j_ϕ^* and S_K^* and cataloguing some of their basic properties. As before, F, H and G always denote (possibly nonabelian) locally compact groups.

- Let $\alpha \in \widehat{F}^1$ and define $M_\alpha : LUC(F) \to LUC(F) : f \mapsto \alpha f$,

$$A_\alpha = M_\alpha^* : LUC(F)^* \to LUC(F)^* : m \mapsto \alpha m.$$

- Let K be a compact normal subgroup of H and define

$$S_K : LUC(H) \to LUC(H/K)$$

by

$$S_K f(xK) = \int_K f(xk)\, dm_K(k), \quad f \in LUC(H),\ xK \in H/K.$$

- Let H be a closed subgroup of G, and let $R_H : LUC(G) \to LUC(H) : f \mapsto f|_H$ be the restriction mapping.
- Let $\theta : F \to H$ be a continuous homomorphism,

$$J_\theta : LUC(H) \to LUC(F) : f \mapsto f \circ \theta, \qquad j_\theta : C_0(H) \to LUC(F) : f \mapsto f \circ \theta.$$

Identifying $M(F)$ with its copy $\Theta_F(M(F))$ in $LUC(F)^*$, we will often slightly abuse notation and write

$$j_\theta^* = j_\theta^* \circ \Theta_F : M(F) \to M(H) \quad \text{and} \quad j_\theta^* = j_\theta^* \circ \Theta_F : L^1(F) \to M(H);$$

by specifying our domains, our intended meaning of j_θ^* should always be clear.

Proposition 5.1. *Let $\theta : F \to H$ be a continuous homomorphism, let φ be either the map $J_\theta^* : LUC(F)^* \to LUC(H)^*$ or $j_\theta^* : LUC(F)^* \to M(H)$, and let $\mathfrak{X}$ denote one of LUC, WAP, E or C_0.*

(i) *φ maps $\mathfrak{X}(F)^*$ into $\mathfrak{X}(H)^*$. When $\mathfrak{X}$ is LUC, WAP, or E, φ is the unique $w^* - w^*$ continuous, contractive, positive homomorphism satisfying $\varphi(\delta_x) = \delta_{\theta(x)}$ $(x \in F)$; $\varphi : M(F) \to M(H)$ is the unique $so_l - w^*$ and $so_r - w^*$ continuous, contractive, positive homomorphism satisfying $\varphi(\delta_x) = \delta_{\theta(x)}$ $(x \in F)$.*
(ii) *The following statements are equivalent*:
  (a) *φ is $w^* - w^*$ continuous on $M(F)$;*
  (b) *θ is a proper map*; *and*
  (c) *φ maps $C_0(F)^\perp$ into $C_0(H)^\perp$.*
(iii) *If φ maps $\mathfrak{X}(F)^*$ onto $\mathfrak{X}(H)^*$, then θ has dense range. When $\mathfrak{X} = LUC$, WAP, or E, and when $\mathfrak{X} = C_0$ and θ is a proper map, the converse holds.*
(iv) *φ maps $L^1(F)$ into $L^1(H)$ if and only if θ is an open mapping.*

Proof. As $j_\theta^* = I_H^* \circ J_\theta^*$ where $I_H : C_0(H) \hookrightarrow LUC(H)$, we will only prove this for $\varphi = J_\theta$.

(i) As $l_x(J_\theta(f)) = J_\theta(l_{\theta(x)} f)$, one can quickly check that J_θ maps $\mathfrak{X}(H)$ into $\mathfrak{X}(F)$ when $\mathfrak{X}$ is *LUC* or *WAP*. Moreover, it is very easy to see that, J_θ maps $B(H)$ into $B(F)$, and therefore J_θ also maps $E(H)$ into $E(F)$. To show that J_θ^* maps $M(F)$ into $M(H)$ it suffices to show that $J_\theta^*(\mu) \in M(H)$, when μ has compact support, L. We show that $J_\theta^*(\mu) = j_\theta^*(\mu)$ in $LUC(H)^*$ (or more precisely that $J_\theta^*(\Theta_F(\mu)) = \Theta_H(j_\theta^*(\Theta_F(\mu))))$. It is easy to see that $j_\theta^*(\mu)$ is supported on $\theta(L)$. Therefore, if $h \in C_0(H)$ is chosen so that $h \equiv 1$ on $\theta(L)$, then for any $f \in LUC(H)$,

$$\begin{aligned} \langle j_\theta^*(\mu), f \rangle_{LUC^*-LUC} &= \int_{\theta(L)} f \, d\big(j_\theta^*(\mu)\big) = \int_{\theta(L)} f h \, d\big(j_\theta^*(\mu)\big) = \langle j_\theta^*(\mu), f h \rangle_{M-C_0} \\ &= \langle \mu, j_\theta(f h) \rangle_{LUC^*-LUC} = \langle \mu, J_\theta(f) \rangle_{LUC^*-LUC} \\ &= \langle J_\theta^*(\mu), f \rangle_{LUC^*-LUC}. \end{aligned}$$

Hence, $J_\theta^*(\mu) \in M(H)$ as claimed. Observe that $j_\theta^*|_{M(F)} = j_\theta^* \circ \Theta_F$ so the $so_l/so_r - w^*$ continuity of φ on $M(F)$ follows from Lemma 1.2. Clearly, $\varphi(m) \geqslant 0$ whenever $m \geqslant 0$, so φ is positive. A quick calculation shows that for $x \in F$, $\varphi(\delta_x) = \delta_{\theta(x)}$ so the remainder of the second statement is a consequence of Lemma 1.1.

(ii) It is not difficult to show (and is well known) that $\theta : F \to H$ is proper if and only if J_θ maps $C_0(H)$ into $C_0(F)$. The equivalence of statements (a) and (b), and the implication (b) implies (c) follows. Conversely, if θ is not proper, there exists $f \in C_0(H)$ such that $J_\theta(f) \notin C_0(F)$. It follows from the Bipolar theorem that there exists some $m \in C_0(F)^\perp$ such that $\langle \varphi(m), f\rangle = \langle m, J_\theta(f)\rangle \neq 0$; hence $\varphi(m) \notin C_0(H)^\perp$.

(iii) Suppose that there is some $h \in H \backslash \overline{\theta(F)}$. If $f \in C_0(H)$ is chosen so that f vanishes on $\theta(F)$ and $f(h) = 1$, then for any $m \in \mathfrak{X}(F)^*$

$$\bigl\langle \varphi(m), f\bigr\rangle = \langle m, f \circ \theta\rangle = m(0) \neq 1 = \langle \delta_h, f\rangle.$$

Hence δ_h is not in the range of φ. Conversely suppose $\overline{\theta(F)} = H$. Then for $h \in H$, $\delta_h \in \overline{\Delta_{\theta(F)}}^{\mathrm{wk}^*}$, so wk*-density of $(\mathbb{C}H)_{\|\cdot\|\leqslant 1}$ in $(\mathfrak{X}(H)^*)_{\|\cdot\|\leqslant 1}$ implies that $(\mathbb{C}\theta(F))_{\|\cdot\|\leqslant 1}$ is also wk*-dense in $(\mathfrak{X}(H)^*)_{\|\cdot\|\leqslant 1}$. Hence, given $m \in \mathfrak{X}(H)^*$ with $\|m\| \leqslant 1$, there is a net $(q_i)_i$ in $(\mathbb{C}\theta(F))_{\|\cdot\|\leqslant 1}$ which converges to m in the wk* topology. Observe that if $q_i = \sum \lambda_j \delta_{\theta(x_j)}$, then $\varphi(p_i) = q_i$ and $\|p_i\| = \|q_i\| \leqslant 1$ where $p_i = \sum \lambda_j \delta_{x_j}$. Passing to a subnet if necessary, we may assume that (p_i) converges wk* to $n \in \mathfrak{X}(F)^*$. Then $\varphi(n) = w^* - \lim \varphi(p_i) = w^* - \lim q_i = m$.

(iv) Suppose first that J_θ^* maps $L^1(F)$ into $L^1(H)$ and let $\{\pi, \mathcal{H}_\pi\} \in \Sigma(H)$, $\xi, \eta \in \mathcal{H}_\pi$. Note that by Eq. (1.1),

$$\bigl\langle \pi(h)\xi \,\big|\, \eta\bigr\rangle = \langle h, \xi *_\pi \eta\rangle_{LUC^*-LUC(H)} \quad \bigl(h \in L^1(H)\bigr),$$

and that $\{\pi \circ \theta, \mathcal{H}_\pi\} \in \Sigma(F)$. Therefore, for $f \in L^1(F)$,

$$\begin{aligned}\bigl\langle \pi\bigl(J_\theta^*(f)\bigr)\xi \,\big|\, \eta\bigr\rangle &= \bigl\langle J_\theta^*(f), \xi *_\pi \eta\bigr\rangle_{LUC^*-LUC(H)} = \langle f, \xi *_{\pi\circ\theta} \eta\rangle_{LUC^*-LUC(F)} \\ &= \bigl\langle (\pi \circ \theta)(f)\xi \,\big|\, \eta\bigr\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\bigl\|J_\theta^*(f)\bigr\|_{C^*(H)} &= \sup\bigl\{\bigl\|\pi\bigl(J_\theta^*(f)\bigr)\bigr\|: \pi \in \Sigma(H)\bigr\} = \sup\bigl\{\bigl\|(\pi \circ \theta)(f)\bigr\|: \pi \in \Sigma(H)\bigr\} \\ &\leqslant \sup\bigl\{\bigl\|\sigma(f)\bigr\|: \sigma \in \Sigma(F)\bigr\} = \|f\|_{C^*(F)}.\end{aligned}$$

It follows that $J_\theta^*|_{L^1(F)} : L^1(F) \to L^1(H)$ extends to a continuous map $\Phi : C^*(F) \to C^*(H)$. Hence, $\Phi^* : B(H) \to B(F)$ is $w^* - w^*$ continuous and a quick calculation shows that $\Phi^*(u) = u \circ \theta$ $(u \in B(H))$. By [23, Proposition 4.3], θ is an open map.

Conversely, suppose that $\theta : F \to H$ is an open continuous homomorphism and let $H_0 = \theta(F)$, $N = \ker\theta$. Then $\theta = \iota_0 \circ \theta_N \circ q_N$ where $q_N : F \to F/N$, $\iota_0 : H_0 \hookrightarrow H$, and $\theta_N : F/N \to H_0$ is a topological isomorphism. It follows that $J_\theta^* = J_{\iota_0}^* \circ J_{\theta_N}^* \circ J_{q_N}^*$. Assuming that Haar measures are normalized so that the Weil formula

$$\int_F f(x)\,dx = \int_{F/N}\int_N f(xn)\,dn\,d(xN) \quad \bigl(f \in L^1(F)\bigr)$$

holds, it is easy to see that on $L^1(F)$, $J^*_{q_N}$ is the canonical map $T_N : L^1(F) \to L^1(F/N)$ given by $T_N f(xN) = \int_N f(xn)\,dn$. As θ_N is a topological isomorphism, $J^*_{\theta_N} f = f \circ \theta_N^{-1} \in L^1(H_0)$ whenever $f \in L^1(F/N)$, and for $f \in L^1(H_0)$, $J^*_{\iota_0} f = f^\circ \in L^1(H)$, where $f^\circ(x) = f(x)$ on H_0, zero elsewhere. Hence, $J^*_\theta = J^*_{\iota_0} \circ J^*_{\theta_N} \circ J^*_{q_N}$ maps $L^1(F)$ into $L^1(H)$. □

We remark that we showed that $J^*_\theta|_{M(F)} = j^*_\theta|_{M(F)}$. As such, in Section 6 we use J_θ interchangeably with j_θ when it is convenient for us to do so. We will often need the following lemma, which follows immediately from Greenleaf's work.

Lemma 5.2. *Let $\iota \in M(H)$. Then ι is a norm-one idempotent lying in the centre of $M(H)$ if and only if $\iota = \rho m_K$ where K is a compact normal subgroup of H and $\rho \in \widehat{K}^1$ is such that $K_0 = \ker\rho$ is normal in H and K/K_0 is contained in the centre of H/K_0.*

Proof. If $\iota = \rho m_K$ with K and ρ as described in the lemma, then $\rho m_K * \delta_x * \rho m_K = \delta_x * \rho m_K$ $(x \in H)$ by [17, Proposition 3.1.6]. A similar calculation to that found in the first paragraph of the proof of [17, Proposition 3.1.6] also shows that $\rho m_K * \delta_x * \rho m_K = \rho m_K * \delta_x$ $(x \in H)$. Hence, ρm_K commutes with δ_x for each $x \in H$ from which it follows that ρm_K is central in $M(H)$. The converse is an immediate consequence of [17, Theorem 2.1.4 and Proposition 3.1.6]. □

Proposition 5.3. *Let $\mathfrak{X}$ denote one of LUC, WAP, E or C_0, let K be a compact normal subgroup of H, H a closed subgroup of G, $\alpha \in \widehat{F}^1$.*

(i) *$A_\alpha : \mathfrak{X}(F)^* \to \mathfrak{X}(F)^*$ is the unique surjective $w^* - w^*$ continuous isometric algebra isomorphism such that $A_\alpha(\delta_x) = \alpha(x)\delta_x$ $(x \in F)$, and $A_\alpha(m_K) = \alpha\, m_K$.*
(ii) *$S^*_K : \mathfrak{X}(H/K)^* \to \mathfrak{X}(H)^*$ is the unique $w^* - w^*$ continuous isometric algebra isomorphism such that $S^*_K(\delta_{xK}) = \delta_x * m_K$ $(xK \in H/K)$.*
(iii) *$R^*_H : \mathfrak{X}(H)^* \hookrightarrow \mathfrak{X}(G)^*$ is the unique $w^* - w^*$ continuous algebra homomorphism such that $R^*_H(\delta_h) = \delta_h$ $(h \in H)$. As a mapping of $M(H)$ into $M(G)$, R^*_H is isometric.*

Proof. (i) Note that $\alpha \in B(F) \subseteq E(F)$, so this is clear. (ii) That S_K maps $C_0(H)$ onto $C_0(H/K)$ is [19, Theorem (15.21)] (for this, compactness of K is not required). It likely is also well known that S_K maps $LUC(H)$ into $LUC(H/K)$ and $WAP(H)$ into $WAP(H/K)$. Indeed, this follows from the identity $l_{xK} S_K f = S_K(l_x f)$ $(x \in H)$ and routine arguments. By [10, Corollary 3.4], S_K maps $B(H)$ into $B(H/K)$, and therefore $E(H)$ into $E(H/K)$. If $f \in \mathfrak{X}(H/K)$, then $f \circ q_K \in \mathfrak{X}(H)$ – for example, by Proposition 5.1 – and $S_K(f \circ q_K) = f$, so S_K is a quotient map and S^*_K is an isometry. It follows from the definition of S_K that $S^*_K(\delta_{xK}) = \delta_x * m_K$. By Lemmas 5.2 and 1.1, S^*_K is a homomorphism. (iii) As $R_H = j_\iota$ where $\iota : H \hookrightarrow G$, most of this follows from Proposition 5.1. That R_H is a quotient mapping of $C_0(G)$ onto $C_0(H)$ is a consequence of the Tietze extension theorem. □

We will write $M(H) \hookrightarrow M(G)$ when we wish to identify $M(H)$ with its w^*-continuous, isometric image $R^*_H(M(H)) = \{\mu \in M(G)\colon\ \operatorname{supp}\mu \subseteq H\}$ in $M(G)$. Note that although R_H maps $LUC(G)$ contractively onto $LUC(H)$ [12], it is not clear to the author that this is a quotient mapping. Moreover, Milnes has shown that $WAP(G)|_H \neq WAP(H)$ and $E(G)|_H \neq E(H)$ [32, example, p. 501].

Here is an example that confirms Kerlin's and Pepe's suspicion that a contractive homomorphism $\varphi : L^1(F) \to M(G)$ may fail to have a Cohen factorization (5.1), and may also fail to have a factorization of the form described in [27, Theorem 1(i)].

Example 5.4. Let K be a compact normal subgroup of F and suppose that $\rho \in \widehat{K}^1$ is chosen so that $\ker\rho$ is also normal in F and $K/\ker\rho$ is contained in the centre of $F/\ker\rho$. By Lemma 5.2, ρm_K is a central norm one idempotent in $M(F)$, and therefore the map

$$\varphi : M(F) \to M(F) : \mu \mapsto \mu * \rho m_K$$

is a contractive $w^* - w^*$, and $so - w^*$, continuous homomorphism. Suppose that ρ does not have a continuous extension to F. (For a specific example of this scenario, one can take $F = SU_2(\mathbb{C})$. Then $Z(F)$, the centre of F, is $\mathbb{Z}_2$ and $\rho : \mathbb{Z}_2 \to \mathbb{T} : t \mapsto t$ does not extend continuously to F [18, p. 322].) Then θ does not have a Cohen factorization and it also does not have a factorization of the form described in [27, Theorem 1(i)].

Indeed, suppose that φ has a Cohen factorization $A_\gamma \circ S_L^* \circ j_\theta^* \circ A_\alpha$ where L is a compact normal subgroup of a closed subgroup H of F, $\theta : F \to H/L$ is a continuous homomorphism, $\alpha \in \widehat{F}^1$, and $\gamma \in \widehat{H}^1$. Observe that because the support subgroup of $\varphi(\Delta_F) = \{\delta_x * m_K\colon x \in F\}$ is F, we must have $H = F$, whence $\gamma \in \widehat{F}^1$. By Propositions 5.1 and 5.3,

$$\rho m_K = \varphi(\delta_{e_F}) = A_\gamma \circ S_L^* \circ j_\theta^* \circ A_\alpha(\delta_{e_F}) = (\gamma|_L) m_L.$$

A comparison of supports gives $L = K$ and continuity of ρ and γ gives $\gamma|_K = \rho$, a contradiction.

We now show that φ does not have a factorization of type (i) described in Theorem 1 of [27]. Suppose that there is a compact normal subgroup L of a locally compact group H, a continuous epimorphism $\theta : F \to H/L$, a continuous monomorphism $\psi : H \to F$, $\gamma \in \widehat{F}^1$, and $\beta \in \widehat{H}^1$ such that $\varphi = j_\psi^* \circ A_\beta \circ S_L^* \circ j_\theta^* \circ A_\gamma$ where, for the convenience of the reader, we have translated Kerlin's and Pepe's notation into our own. Observe that the map $\psi_L : L \to \psi(L)$ is a topological isomorphism, so $j_\psi^*(\beta m_L) = (\beta \circ \psi_L^{-1}) m_{\psi(L)}$ follows from the identity $\int_L f(\psi_L(l))\, dm_L(l) = \int_{\psi(L)} f(k)\, dm_{\psi(L)}(k)$. Let $k \in K$ and suppose that $\theta(k) = xL$. Then the last observation, together with Propositions 5.1 and 5.3, gives

$$\overline{\rho(k)}\rho m_K = \varphi(\delta_k) = j_\psi^* \circ A_\beta \circ S_L^* \circ j_\theta^* \circ A_\gamma(\delta_k) = \gamma(k)\beta(x)\delta_{\psi(x)} * \big(\beta \circ \psi_L^{-1}\big) m_{\psi(L)}.$$

When $k = e_F$, this becomes $\rho m_K = (\beta \circ \psi_L^{-1}) m_{\psi(L)}$, so $K = \psi(L)$ and $\rho = \beta \circ \psi_L^{-1}$. Hence the last equation, in general, becomes

$$\overline{\rho(k)}\rho m_K = \gamma(k)\beta(x)\delta_{\psi(x)} * \rho m_K. \tag{5.2}$$

Comparing supports gives $\psi(x) \in K$ whenever $\theta(k) = xL$. As ψ is one-to-one and maps L onto K we have $x \in L$ and therefore $\theta(k) = xL = L$. We can therefore replace x in (5.2) by e_H to obtain $\overline{\rho(k)}\rho m_K = \gamma(k)\rho m_K$ $(k \in K)$. Hence, $\overline{\gamma} \in \widehat{F}^1$ is an extension of ρ, in contradiction to our assumptions.

5.2. Factorizations of contractive homomorphisms

We begin by noting that our earlier discussion of $*$-homomorphisms applies to the contractive case.

Proposition 5.5. *Every $so_l - w^*$ continuous, contractive homomorphism $\varphi : M(F) \to M(G)$ is a $*$-homomorphism. Moreover, if $\Gamma = \varphi(\Delta_F)$ has support subgroup H and $\phi : F \to \Gamma : x \mapsto \varphi(\delta_x)$, then $\varphi = \kappa_\phi^* : M(F) \to M(H) \hookrightarrow M(G)$.*

Proof. Note that $M(H)$, the range of R_H^* in $M(G)$, is norm-closed, and is therefore also w^*-closed in $M(G)$. As $\Gamma = \varphi(\Delta_F) \subseteq M(H)$ and φ is $so_l - w^*$ continuous, $\varphi(M(F)) = \varphi(\overline{\mathbb{C}\Delta_F}^{so_l}) \subseteq \overline{\varphi(\mathbb{C}\Delta_F)}^{w^*} \subseteq M(H)$. Observe that by Lemma 4.1, it makes sense to speak of $\kappa_\phi : C_0(H) \to E(F)$. For $x \in F$, $\kappa_\phi^*(\delta_x) = \phi(x) = \varphi(\delta_x)$ so by Lemma 1.1, the $so_l - w^*$ continuity of both maps on $M(F)$ gives $\varphi = \kappa_\phi^* \circ \Theta_F$. By Theorem 3.8, φ is a $*$-homomorphism. □

Let $\varphi : L^1(F) \to M(G)$ be a contractive homomorphism, φ_m its extension to $M(F)$. We will now prove Theorem 5.7 which shows how φ_m factors as the product of a contractive, positive, $so_l - w^*$ continuous $*$-homomorphism $j_\theta^* : M(F) \to M(Q) : \delta_x \mapsto \delta_{\theta(x)}$ and a $w^* - w^*$ continuous contractive $*$-homomorphism $\kappa_\phi^* : M(Q) \to M(G) : \delta_{(\alpha,t)\Omega_\rho} \mapsto \alpha\delta_t * \rho m_K$; here Q can be either one of the locally compact groups $\overline{\Omega}/\Omega_\rho$ or $\mathbb{T} \times H/\Omega_\rho$ associated with $\Gamma = \phi_m(\Delta_F)$. Theorem 5.7 hence improves the main characterizations of contractive homomorphisms from [17], namely Theorems 4.2.1, 4.2.2 and 4.2.3. Later, we shall improve this further.

Lemma 5.6. *Suppose that Γ is a contractive subgroup of $M(G)$ for which there is a locally compact group Q and a topological isomorphism $\phi : Q \to \Gamma$. Then $\kappa_\phi : C_0(G) \to E(Q)$ maps into $C_0(Q)$.*

Proof. By Theorem 4.2, $\phi_0 : \Omega/\Omega_\rho \to \Gamma : (\alpha, t)\Omega_\rho \mapsto \alpha\delta_t * \rho m_K$ is a topological isomorphism. Hence, $\theta = \phi_0^{-1} \circ \phi$ is a topological isomorphism of Q onto Ω/Ω_ρ and therefore, by Proposition 5.1, $J_\theta : E(\Omega/\Omega_\rho) \to E(Q)$ maps $C_0(\Omega/\Omega_\rho)$ onto $C_0(Q)$. As $J_\theta \circ \kappa_{\phi_0} = \kappa_\phi$, it hence suffices to show that κ_{ϕ_0} maps $C_0(G)$ into $C_0(\Omega/\Omega_\rho)$. To this end, take $u \in C_{00}(G)$ with compact support L, and put $S_L = \{x \in \Omega/\Omega_\rho\colon \operatorname{supp}(\phi_0(x)) \cap L \neq \emptyset\}$. Observe that if $x \in (\Omega/\Omega_\rho)\backslash S_L$, then $\kappa_{\phi_0}u(x) = \int_L u \, d\phi_0(x) = 0$, so $\operatorname{supp}(\kappa_{\phi_0}u) \subseteq \overline{S_L}$. If $x = (\alpha, t)\Omega_\rho \in S_L$, then $\operatorname{supp}(\phi_0(x)) \cap L = tK \cap L \neq \emptyset$, so $t \in LK$. Hence, S_L is contained in $(\mathbb{T} \times LK/\Omega_\rho) \cap \Omega/\Omega_\rho$, a compact subset of the locally compact group – in this case – Ω/Ω_ρ. □

Theorem 5.7. *Let $\varphi : M(F) \to M(G)$ be an $so_l - w^*$ continuous contractive homomorphism. Let Γ denote the contractive subgroup $\varphi(\Delta_F)$ of $M(G)$ with support group H, $\phi : (\alpha, t)\Omega_\rho \mapsto \alpha\delta_t * \rho m_K$ the topological isomorphism of locally compact groups $\overline{\Omega}/\Omega_\rho$ onto $\Gamma_{\overline{\Omega}}$, and $\mathbb{T} \times H/\Omega_\rho$ onto $\Gamma_{\mathbb{T}\times H}$. Then*:

- (i) *there exists a continuous homomorphism of locally compact groups with dense range, $\theta : F \to \overline{\Omega}/\Omega_\rho$ such that $\varphi = \kappa_\phi^* \circ j_\theta^*$; and*
- (ii) *there exists a continuous homomorphism of locally compact groups $\theta : F \to \mathbb{T} \times H/\Omega_\rho$ such that $\varphi = \kappa_\phi^* \circ j_\theta^*$.*

Proof. Let $Q = \overline{\Omega}/\Omega_\rho$, $\phi : Q \to \Gamma_{\overline{\Omega}}$, and define $\theta : F \to Q$ by

$$\theta(x) = \phi^{-1}\big(\varphi(\delta_x)\big) \quad (x \in F).$$

As $x \mapsto \delta_x : F \to (\Delta_F, so_l)$ is a topological isomorphism, $so_l - w^*$ continuity φ and continuity of ϕ^{-1} imply that θ is a continuous homomorphism with image $\theta(F) = \Omega/\Omega_\rho$, a dense subgroup of Q. By Proposition 5.1, $j_\theta^* : M(F) \to M(Q)$ is $so_l - w^*$ continuous and, by Lemma 5.6, $\kappa_\phi^* : M(Q) \to M(G)$ is $w^* - w^*$ continuous. Hence, $\kappa_\phi^* \circ j_\theta^*$ is an $so_l - w^*$ continuous homomorphism satisfying

$$\kappa_\phi^* \circ j_\theta^*(\delta_x) = \kappa_\phi^*(\delta_{\theta(x)}) = \phi\big(\theta(x)\big) = \varphi(\delta_x) \quad (x \in F).$$

By Lemma 1.1, $\varphi = \kappa_\phi^* \circ j_\theta^*$, proving (i). The same argument, with $Q = \mathbb{T} \times H/\Omega_\rho$ and $\phi : Q \to \Gamma_{\mathbb{T}\times H}$, proves (ii). $\square$

The following corollary slightly extends one of the main results in [27]. In particular, the result shows that a contractive homomorphism has a Cohen factorization whenever its support subgroup is abelian.

Corollary 5.8. *Let $\varphi : M(F) \to M(G)$ be an $so_l - \mathrm{wk}^*$ continuous homomorphism with $\varphi(\delta_{e_F}) = \rho m_K$ and support subgroup H of $\Gamma = \varphi(\Delta_F)$. If ρ extends to $\rho_H \in \widehat{H}^1$, then there exists some $\alpha \in \widehat{F}^1$ and a continuous homomorphism $\theta_K : F \to H/K$ such that $\varphi : M(F) \to M(H) \hookrightarrow M(G)$ has Cohen factorization $\varphi = A_{\rho_H} \circ S_K^* \circ j_{\theta_K}^* \circ A_\alpha$.*

Proof. Let $\theta : F \to \mathbb{T} \times H/\Omega_\rho$ and $\phi : \mathbb{T} \times H/\Omega_\rho \to \Gamma_{\mathbb{T}\times H}$ be such that $\varphi = \kappa_\phi^* \circ j_\theta^*$ as in Theorem 5.7. It is easy to see that the maps

$$p_{\mathbb{T}} : \mathbb{T} \times H/\Omega_\rho \to \mathbb{T} : (\alpha, h)\Omega_\rho \mapsto \alpha\overline{\rho_H(h)}, \qquad p_K : \mathbb{T} \times H/\Omega_\rho \to H/K : (\alpha, h)\Omega_\rho \mapsto hK$$

are well-defined continuous homomorphisms. Therefore $\alpha = p_{\mathbb{T}} \circ \theta \in \widehat{F}^1$ and $\theta_K = p_K \circ \theta : F \to H/K$ is a continuous homomorphism. Suppose that $\theta(x) = (\gamma_x, h_x)\Omega_\rho$, so $\varphi(\delta_x) = \gamma_x\delta_{h_x} * \rho m_K$. As $\alpha(x) = \gamma_x\overline{\rho_H(h_x)}$ and $\theta_K(x) = h_x K$,

$$\begin{aligned} A_{\rho_H} \circ S_K^* \circ j_{\theta_K}^* \circ A_\alpha(\delta_x) &= A_{\rho_H} \circ S_K^*\big(\gamma_x\overline{\rho_H(h_x)}\delta_{h_x K}\big) = \gamma_x\overline{\rho_H(h_x)}A_{\rho_H}(\delta_{h_x} * m_K) \\ &= \gamma_x\overline{\rho_H(h_x)}\rho_H(h_x)\delta_{h_x} * \rho m_K = \varphi(\delta_x). \end{aligned}$$

By Lemma 1.1, φ has the desired Cohen factorization. $\square$

We will now prove our main theorem which provides a very simple description of any contractive homomorphism as a product of four canonically defined homomorphisms.

- Let

$$\theta_H : \mathbb{T} \times H \to H : (\alpha, h) \mapsto h \quad \text{and} \quad \alpha_{\mathbb{T}} : \mathbb{T} \times H \to \mathbb{T} : (\alpha, h) \mapsto \alpha$$

be the projection homomorphism and the projection character on $\mathbb{T} \times H$. Let

$$\Lambda_H = M_{\alpha_{\mathbb{T}}} \circ j_{\theta_H} : CB(H) \to CB(\mathbb{T} \times H), \qquad \Lambda_H f(\alpha, h) = \alpha f(h), \quad (\alpha, h) \in \mathbb{T} \times H.$$

Proposition 5.9. *Let $\mathfrak{X}$ denote LUC, WAP, E or C_0. Then $\Lambda_H = M_{\alpha_{\mathbb{T}}} \circ j_{\theta_H}$ is a linear isometry mapping $\mathfrak{X}(H)$ into $\mathfrak{X}(\mathbb{T} \times H)$ and $\Lambda_H^* = j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} : \mathfrak{X}(\mathbb{T} \times H)^* \to \mathfrak{X}(H)^*$ is a contractive $w^* - w^*$ continuous surjective homomorphism such that $\Lambda_H^* \delta_{(\alpha,h)} = \alpha \delta_h$ and, for any compact subgroup K of H and any $\rho \in \widehat{K}^1$, $\Lambda_H^* m_{\Omega_\rho} = \rho m_K$.*

Proof. Most of this follows immediately from Propositions 5.1 and 5.3. To see that $\Lambda_H^* m_{\Omega_\rho} = \rho m_K$, first note that $\gamma : K \to \Omega_\rho : k \mapsto (\rho(k), k)$ is a topological isomorphism of compact groups, so $j_\gamma^*(m_K) = m_{\Omega_\rho}$, where $j_\gamma^* : M(K) \to M(\Omega_\rho)$; $j_\gamma^* m_K$ is positive, translation invariant, and $\|j_\gamma^* m_K\| = 1$. Hence, for $f \in C_0(H)$,

$$\begin{aligned} \langle \Lambda_H^* m_{\Omega_\rho}, f \rangle = \langle j_\gamma^* m_K, \Lambda_H f \rangle &= \int_K \Lambda_H f\bigl(\gamma(k)\bigr)\, dm_K(k) \\ &= \int_K \Lambda_H f\bigl(\rho(k), k\bigr)\, dm_K(k) = \int_K \rho(k) f(k)\, dm_K(k) \\ &= \langle \rho m_K, f \rangle \end{aligned}$$

as needed. $\square$

Proposition 5.10. *Let Γ be a contractive subgroup of $M(G)$ with identity ρm_K and support subgroup H. If $\phi : \mathbb{T} \times H/\Omega_\rho \to \Gamma_{\mathbb{T}\times H} \subseteq M(H) : (\alpha, t)\Omega_\rho \mapsto \alpha \delta_t * \rho m_K$ is the associated topological isomorphism, then the contractive $w^* - w^*$ continuous homomorphism*

$$\kappa_\phi^* : M(\mathbb{T} \times H/\Omega_\rho) \to M(H)$$

factors as $\kappa_\phi^* = j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^*$.

Proof. By Lemma 5.6 and Propositions 5.3 and 5.9, each of these homomorphisms is $w^* - w^*$ continuous, and

$$j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^* \delta_{(\alpha,t)\Omega_\rho} = \Lambda_H^* (\delta_{(\alpha,t)} * m_{\Omega_\rho}) = \alpha \delta_t * \rho m_K = \phi\bigl((\alpha, t)\Omega_\rho\bigr) = \kappa_\phi^* (\delta_{(\alpha,t)\Omega_\rho}).$$

By Lemma 1.1, $\kappa_\phi^* = j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^*$. $\square$

Theorem 5.11. *Let F and G be locally compact groups, and let $\varphi : L^1(F) \to M(G)$ be a mapping. Then the following statements are equivalent*:

(i) *φ is a contractive homomorphism*;
(ii) *there is a closed subgroup H of G, a compact normal subgroup Ω_0 of $\mathbb{T} \times H$, and a continuous homomorphism $\theta : F \to \mathbb{T} \times H/\Omega_0$ such that the diagram commutes*:

$$\begin{tikzcd}
L^1(F) \arrow[rr, "\varphi"] \arrow[d, "j_\theta^*"'] & & M(H) \arrow[r, hook] & M(G) \\
M(\mathbb{T}\times H/\Omega_0) \arrow[r, "S_{\Omega_0}^*"] & M(\mathbb{T}\times H) \arrow[r, "A_{\alpha_{\mathbb{T}}}"] & M(\mathbb{T}\times H) \arrow[u, "j_{\theta_H}^*"'] &
\end{tikzcd}$$

That is, φ factors as

$$\varphi = j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_0}^* \circ j_\theta^*.$$

In statement (ii) *we may take $\Omega_0 = \Omega_\rho$ for some compact normal subgroup K of H and $\rho \in \widehat{K}^1$ such that* $\ker\rho$ *is normal in H and $K/\ker\rho$ lies in the centre of $H/\ker\rho$.*

Proof. Let $\varphi_m : M(F) \to M(G)$ be the $so_l - w^*$ continuous extension of φ to $M(F)$. By Proposition 5.5, Theorem 5.7, and Proposition 5.10, φ_m (and therefore φ) has the desired factorization. As $H = \overline{H_0}$, the final statement follows from [17, Theorem 3.1.8]. □

Remarks 5.12.

1. Note that the contractive homomorphisms j_θ^*, $S_{\Omega_0}^*$, $A_{\alpha_{\mathbb{T}}}$ and $j_{\theta_H}^*$ have simple descriptions and are very well-understood. Moreover, each factor is of one of the three basic types appearing in the Cohen factorization.
2. The product $\Lambda_H^* = j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}}$ is universal in the sense that it appears as a factor of φ whenever the support subgroup of $\Gamma = \varphi_m(\Delta_F)$ equals H.
3. Observe that if $\theta_{H,G} : \mathbb{T}\times H \to G : (\alpha, h) \mapsto h$ and, as before, $R_H^* : M(H) \hookrightarrow M(G)$, then $R_H^* \circ j_{\theta_H}^* = j_{\theta_{H,G}}^*$.

6. Various corollaries

Suppose now that $\varphi : L^1(F) \to M(H) \xrightarrow{R_H^*} M(G)$ has the factorization $\varphi = R_H^* \circ j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^* \circ j_\theta^*$ described in Theorem 5.11. By Propositions 5.1, 5.3, and 5.9, when $\mathfrak{X}$ is *LUC*, *WAP*, or E (resp. $\mathfrak{X} = C_0$)

$$R_H^* \circ j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^* \circ J_\theta^* : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^* \tag{6.1}$$

and

$$R_H^* \circ j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^* \circ j_\theta^* : \mathfrak{X}(F)^* \to M(G) \tag{6.2}$$

define $w^* - w^*$ (resp. $so - w^*$) continuous, contractive $*$-homomorphisms. We will refer to factorizations of homomorphisms $\mathfrak{X}(F)^* \to \mathfrak{X}(G)^*$, $\mathfrak{X}(F)^* \to M(G)$ and $M(F) \to M(G)$ of the form (6.1) or (6.2) as *canonical factorizations.* As an immediate consequence of Theorem 5.11 and the w^* (resp. so_l) density of $L^1(F)$ in $\mathfrak{X}(F)^*$ (resp. $M(F)$), we have the following statement.

Corollary 6.1. *Let $\mathfrak{X}$ denote one of LUC, WAP, or E. Every contractive $w^* - w^*$ (resp. $so_l - w^*$) continuous, contractive homomorphism $\varphi : \mathfrak{X}(F)^* \to M(G)$ (resp. $\varphi : M(F) \to M(G)$) has a canonical factorization.*

Observe that by Propositions 5.1, 5.3 and 5.9, homomorphisms $\mathfrak{X}(F)^* \to X(G)^*$ with canonical factorizations always map $M(F)$ into $M(G)$; here $\mathfrak{X}(H)^* = M(H) \oplus_1 C_0(H)^\perp$. A dual version of the following immediate corollary to Theorem 5.11 is proved in [24].

Corollary 6.2. *Let $\mathfrak{X}$ denote one of LUC, WAP, or E. Then every contractive homomorphism $\varphi : L^1(F) \to M(G)$ extends to $w^* - w^*$ continuous contractive homomorphisms*

$$\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^* \quad \textit{and} \quad \varphi : \mathfrak{X}(F)^* \to M(G)$$

extending $\varphi_m : M(F) \to M(G)$.

Corollary 6.3. *Let $\mathfrak{X}$ denote one of LUC, WAP, or E, $\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^*$. Then following statements are equivalent*:

(i) *φ is a $w^* - w^*$ continuous homomorphism such that φ is contractive on Δ_F and $\varphi(\mu_0) \notin C_0(G)^\perp$ for some $\mu_0 \in M(F)$;*
(ii) *φ has a canonical factorization $\varphi = R_H^* \circ j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^* \circ J_\theta^*$.*

Proof. Assume that condition (i) holds and consider the $w^* - w^*$ continuous projection

$$\mathfrak{R}_G : \mathfrak{X}(G)^* = M(G) \oplus_1 C_0(G)^\perp \to M(G) : \mu \oplus \nu \mapsto \mu.$$

The map $\varphi_M = \mathfrak{R}_G \circ \varphi \circ \Theta_F : M(F) \to M(G)$ is an $so - w^*$ continuous homomorphism of $M(F)$ which, because $\varphi(\mu_0) \notin C_0(G)^\perp$, is non-zero. Letting $\Gamma = \varphi_M(\Delta_F)$, Γ is a contractive subgroup of $M(G)$ which, by Lemma 1.1, is non-zero. If $\phi : F \to \Gamma : x \mapsto \varphi_M(\delta_x)$, then $\kappa_\phi^* : E(F)^* \to M(G) : \delta_x \mapsto \varphi_M(\delta_x)$ so – again by Lemma 1.1 – $\varphi_M = \kappa_\phi^* \circ \Theta_F$. Hence, φ_M is contractive and therefore has a canonical factorization $\varphi_M = R_H^* \circ j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^* \circ J_\theta^*$ by Corollary 6.1. Let $x \in F$ and write $\varphi(\delta_x) = \mu_x + \nu_x$ where $\mu_x \in M(G)$, $\nu_x \in C_0(G)^\perp$ and $\|\varphi(\delta_x)\| = \|\mu_x\| + \|\nu_x\|$. Then $\mu_x = \varphi_M(\delta_x) \in \Gamma$ and therefore,

$$1 = \|\mu_x\| \leqslant \|\mu_x\| + \|\nu_x\| = \big\|\varphi(\delta_x)\big\| \leqslant 1;$$

hence, $\varphi(\delta_x) = \mu_x = \varphi_M(\delta_x) \in M(G)$. If $\varphi_c : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^*$ is the $w^* - w^*$ continuous homomorphism with canonical factorization $\varphi_c = R_H^* \circ j_{\theta_H}^* \circ A_{\alpha_{\mathbb{T}}} \circ S_{\Omega_\rho}^* \circ J_\theta^*$, we now have $\varphi_c(\delta_x) = \varphi_M(\delta_x) = \varphi(\delta_x)$ $(x \in F)$ so $\varphi = \varphi_c$ by Lemma 1.1. The converse implication is trivial. $\square$

One of the main results from [14] states that if φ is an isometric isomorphism of $LUC(F)^*$ onto $LUC(G)^*$, then φ maps $M(F)$ into $M(G)$. The next corollary follows immediately from Corollary 6.3.

Corollary 6.4. *Let $\mathfrak{X}$ denote one of LUC, WAP, or E, and let $\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^*$ be a contractive $w^* - w^*$ continuous homomorphism. If $\varphi(\mu_0) \notin C_0(G)^\perp$ for some $\mu_0 \in M(F)$, then φ maps $M(F)$ into $M(G)$.*

Example 6.5. Let F be a non-compact locally compact group, $\mathfrak{X}(F)$ a translation-invariant subspace of $WAP(F)$ which contains 1_F and $C_0(F)$. If F is further assumed to be amenable, then we may take $\mathfrak{X}(F) = LUC(F)$. In each of these cases, there exists a non-zero contractive $w^* - w^*$ continuous homomorphism $\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(F)^*$ such that φ maps $\mathfrak{X}(F)^*$ into $C_0(F)^\perp$. Hence, the condition that $\varphi(\mu_0) \notin C_0(G)^\perp$ for some $\mu_0 \in M(F)$ was essential to Corollary 6.4.

To see this, first note that in all of the above cases there is a left invariant mean $m \in \mathfrak{X}(F)^*$; m is a positive, norm one functional such that $m(f \cdot x) = m(f)$ for $f \in \mathfrak{X}(F)$ and $x \in F$. Define

$$\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(F)^* : n \mapsto n(1_F)m.$$

It is well known, and not difficult to check, that $m \in C_0(F)^\perp$ and $m * m = m$. As $(n_1 * n_2)(1_F) = n_1(1_F)n_2(1_F)$, it readily follows that φ has the desired properties.

Corollary 6.6. *Let $\mathfrak{X}$ denote one of LUC, WAP, or E and let*

$$\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^* \quad \text{or} \quad \varphi : \mathfrak{X}(F)^* \to M(G),$$

be a $w^* - w^*$ continuous (resp. let $\varphi : M(F) \to M(G)$ be an $so_l - w^*$ continuous) contractive homomorphism such that $\varphi(\mu_0) \notin C_0(G)^\perp$ for some $\mu_0 \in M(F)$. Let $\Gamma = \varphi(\Delta_F) \subseteq M(G)$ have support subgroup H.

(i) *If $\varphi(\delta_{e_F}) = \rho m_K$ and ρ extends to some $\rho_H \in \widehat{H}^1$, then φ has a Cohen factorization $\varphi = R_H^* \circ A_{\rho_H} \circ S_K^* \circ J_{\theta_K}^* \circ A_\alpha$.*
(ii) *$\varphi(\delta_{e_F}) \geqslant 0$ if and only if there is a compact normal subgroup K of H, $\alpha \in \widehat{F}^1$ and a continuous homomorphism $\theta : F \to H/K$ such that $\varphi = R_H^* \circ S_K^* \circ J_{\theta_K}^* \circ A_\alpha$.*

Proof. We prove this in the case that $\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^*$. Assume the hypothesis of statement (i). By Corollary 6.4, φ maps $M(F)$ into $M(G)$ so by Corollary 5.8, $\varphi|_{M(F)} (= \mathfrak{R}_G \circ \varphi \circ \Theta_F)$ has Cohen factorization $\varphi|_{M(F)} = R_H^* \circ A_{\rho_H} \circ S_K^* \circ J_{\theta_K}^* \circ A_\alpha$. By Lemma 1.1, (i) follows. If $\varphi(\delta_{e_F}) \geqslant 0$, then by [17, Proposition 2.1.3], $\varphi(\delta_{e_F}) = m_K$; that is, $\rho = 1$. Hence, part (ii) follows from part (i). □

The following important special case of part (ii) of this last corollary is worth recording.

Corollary 6.7. *Let $\mathfrak{X}$ denote one of LUC, WAP, or E and let*

$$\varphi : \mathfrak{X}(F)^* \to \mathfrak{X}(G)^* \quad \text{or} \quad \varphi : \mathfrak{X}(F)^* \to M(G),$$

be a $w^* - w^*$ continuous (resp. let $\varphi : M(F) \to M(G)$ be an $so_l - w^*$ continuous) contractive homomorphism. Then $\varphi(\delta_{e_F}) = \delta_{e_G}$ if and only if there is a continuous homomorphism $\theta : F \to G$ and a character $\alpha \in \widehat{F}^1$ such that $\varphi = J_\theta^* \circ A_\alpha$.

Corollary 6.8. *Let $\mathfrak{X}$ denote one of LUC, WAP, or E and let φ be a $w^* - w^*$ continuous isomorphism of $\mathfrak{X}(F)^*$ onto $\mathfrak{X}(G)^*$ that is contractive on Δ_F. Then there is a topological isomorphism θ mapping F onto G and $\alpha \in \widehat{F}^1$ such that $\varphi = J_\theta^* \circ A_\alpha$. Hence, φ is an isometric $*$-isomorphism mapping $M(F)$, and $L^1(F)$, as an isometric $*$-isomorphism onto $M(G)$, and $L^1(G)$, respectively.*

Proof. Clearly $\varphi(\delta_{e_F}) = \delta_{e_G}$, so by Corollary 6.3 φ is contractive and by Corollary 6.7, there exists $\alpha \in \widehat{F}^1$ and a continuous homomorphism $\theta : F \to G$ such that $\varphi = J_\theta^* \circ A_\alpha$. Observe that φ^{-1} is also necessarily $w^* - w^*$ continuous and contractive on $\Delta_{\theta(F)}$. By Proposition 5.1, $\theta(F)$ is dense in G, so φ^{-1} is contractive on Δ_G. Hence, there exists $\beta \in \widehat{G}^1$ and a continuous homomorphism $\psi : G \to F$ such that $\varphi^{-1} = J_\psi^* \circ A_\beta$. For any $x \in F$, $\delta_x = \varphi^{-1}(\varphi(\delta_x)) = \alpha(x)\beta(\theta(x))\delta_{\psi(\theta(x))}$, so $x = \psi(\theta(x))$. Similarly $\theta(\psi(y)) = y$ ($y \in G$). Hence θ is a topological isomorphism with inverse ψ. The final statement is known (and also follows from Proposition 5.1). □

Let $\varphi : LUC(F)^* \to LUC(G)^*$ be an isometric isomorphism. It is shown in [14] that on $M(F)$, φ agrees with $J_\theta^* \circ A_\alpha$. It would be interesting to know if any such φ is necessarily $w^* - w^*$ continuous on $LUC(F)^*$ and hence of the form $J_\theta^* \circ A_\alpha$.

Corollary 6.9. *Let $(e_i)_i$ be an approximate identity for $L^1(F)$, $\varphi : L^1(F) \to M(G)$ (respectively $\varphi : L^1(F) \to L^1(G)$). Then φ is a contractive homomorphism such that $\delta_{e_G} \in \overline{\{\varphi(e_i)\}}^{w^*}$ if and only if there exists $\alpha \in \widehat{F}^1$ and a continuous homomorphism (respectively a continuous, open homomorphism) $\theta : F \to G$ such that $\varphi = j_\theta^* \circ A_\alpha$.*

Proof. Suppose that φ is a contractive homomorphism such that $\delta_{e_G} \in \overline{\{\varphi(e_i)\}}^{w^*}$ and let $\varphi_m : M(F) \to M(G)$ be the $so_l - w^*$ continuous extension of φ. Passing to a subnet if necessary, we may assume without loss of generality that $w^* - \lim \varphi(e_i) = \delta_{e_G}$. Clearly $so_l - \lim e_i = \delta_{e_F}$, so $\varphi_m(\delta_{e_F}) = w^* - \lim \varphi_m(e_i) = \delta_{e_G}$. By Corollary 6.7 (and Proposition 5.1), φ_m is has the desired form. □

Contractive epimorphisms between group algebras have been characterized by Greenleaf [17] and Kerlin and Pepe [27]. With the following corollary, we give a simple, new characterization of these maps which is independent of the earlier results.

Corollary 6.10. *Let $\varphi : L^1(F) \to L^1(G)$. Then φ is a contractive epimorphism if and only if there exists $\alpha \in \widehat{F}^1$ and a continuous open epimorphism $\theta : F \to G$ such that $\varphi = j_\theta^* \circ A_\alpha$.*

Proof. Let (e_i) be a bounded approximate identity for $L^1(F)$. Assuming that φ is a contractive epimorphism, $(\varphi(e_i))$ is a bounded approximate identity for $L^1(G)$ so by Lemma 1.2, δ_{e_G} is the w^*-limit of $\varphi(e_i)$ in $M(G)$. By Corollary 6.9, $\varphi = j_\theta^* \circ A_\alpha$ for some $\alpha \in \widehat{F}^1$ and some continuous, open homomorphism $\theta : F \to G$. Suppose that there is some y in $G \backslash \theta(F)$, and choose $f \in L^1(F)$ such that $y \in \operatorname{supp}(\varphi(f))$. As $G \backslash \theta(F)$ is open, we can find $\psi \in C_0(G)$ with support contained in $G \backslash \theta(F)$ such that

$$0 \neq \langle \varphi(f), \psi \rangle = \langle f, \alpha(\psi \circ \phi) \rangle = 0,$$

a contradiction. Hence, θ is surjective. Suppose now that φ takes the converse form, and let $N = \ker \theta$. Then, as in the proof of Proposition 5.1(iv), $j_\theta^* = j_{\theta_N}^* \circ T_N$ where $T_N : L^1(F) \to L^1(F/N)$ is the canonical surjection, and $j_{\theta_N}^* : L^1(F/N) \to L^1(G)$ is the isometric isomorphism associated with the topological isomorphism $\theta_N : F/N \to G$. Hence, $\varphi = j_\theta^* \circ A_\alpha$ is a contractive epimorphism. □

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